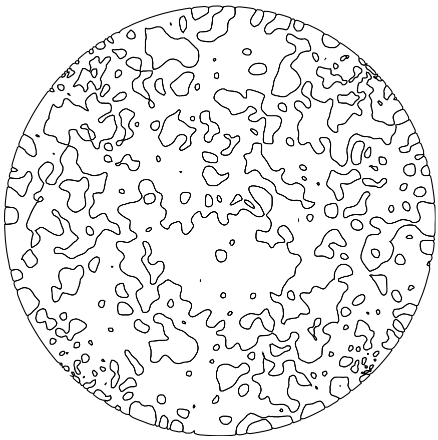
CONTRIBUTIONS TO THE THEORY OF MULTIPLICATIVE CHAOS

JANNE JUNNILA



Academic dissertation

To be presented, with the permission of the Faculty of Science of the University of Helsinki, for public criticism in Auditorium XII of the Main Building of the University on 6th October 2018 at 10 o'clock.

Department of Mathematics and Statistics
Faculty of Science
University of Helsinki

Helsinki 2018

ISBN 978-951-51-4513-0 (paperback)
ISBN 978-951-51-4514-7 (PDF)

http://ethesis.helsinki.fi

Unigrafia Oy
Helsinki 2018

ACKNOWLEDGMENTS

Above all, I am most grateful to my advisor Eero Saksman for his excellent support and guidance during my studies. His vast knowledge of mathematics in general, and analysis in particular, has certainly left me impressed, and hopefully a bit of it will stick with me as well. I have also always liked Eero's habit of telling stories from the history of mathematics. Besides being entertaining, these anecdotes connect the theorems to the human beings behind them and illustrate how diverse the mathematical community actually is.

Next, I would like to give special thanks to Christian Webb for his collaboration and insights. Especially, his ability to place our work into the larger context of mathematical physics has been an illuminating experience for me.

I wish to thank Julien Barral and Vincent Vargas for their efforts and dedication when working as the pre-examiners of this thesis. Their positive comments are highly appreciated.

I also thank the funders of this research: the Doctoral Programme in Mathematics and Statistics at the University of Helsinki, the Academy of Finland Centre of Excellence 'Analysis and Dynamics', as well as the Academy of Finland Project 'Conformal methods in analysis and random geometry'.

My stay at the Kumpula campus has been a joyous time. Thanks here go to all my friends and colleagues for creating the cozy atmosphere. I thank Otte Heinävaara for the numerous interesting discussions we have had, as well as for running the high school math club Tekijäryhmä with me. I also thank my friends in the Finnish competitive programming community for providing me with lots of pastimes, whether it be solving coding problems, playing the violin, or just chatting on the IRC. Likewise, I thank everyone who has played with or against me at the traditional Friday badminton sessions of the math department.

Finally, I wish to thank my mom and dad for all their support, as well as my sister for all the silly laughs!

Helsinki, September 2018 Janne Junnila

LIST OF INCLUDED ARTICLES

This thesis consists of an introduction and the following three articles:

[**I**] J. Junnila and E. Saksman. *Uniqueness of critical Gaussian chaos*. Electronic Journal of Probability 22 (2017).

[**II**] J. Junnila. *On the Multiplicative Chaos of Non-Gaussian Log-Correlated Fields*. International Mathematics Research Notices (2018), rny196.

[**III**] J. Junnila, E. Saksman, and C. Webb. *Imaginary multiplicative chaos: Moments, regularity and connections to the Ising model*. To be submitted (2018).

All respective authors played equal roles in the analysis and writing of the joint articles [**I**] and [**III**].

CONTENTS

1

WHAT IS MULTIPLICATIVE CHAOS?

1.1 *An introductory example*

The subject of this thesis is the study of various instances of random distributions that we collectively call *multiplicative chaos*.[1] They are often constructed by taking the product of an infinite number of independent random functions, which explains the name.

One of the simplest examples of multiplicative chaos is based on the random Fourier series

$$X(x) := \sum_{k=1}^{\infty} \frac{A_k \cos(2\pi k x) + B_k \sin(2\pi k x)}{\sqrt{k}}, \tag{1.1}$$

where A_k and B_k are independent standard Gaussian random variables. Let $Y_n(x)$ denote the nth term in the series (1.1), fix a parameter $\beta > 0$, and define for $n \geq 1$ and $x \in [0,1]$ the product

$$M_n(x) := \prod_{k=1}^{n} \exp\Big(\beta Y_k(x) - \frac{\beta^2}{2} \mathbb{E}\, Y_k(x)^2\Big).$$

It is easy to check that the sequence $(M_n(x))_{n=1}^{\infty}$ is a non-negative martingale, and from this it follows that the measures $M_n(x)\,dx$ converge almost surely in the weak*-sense to a random measure μ on the interval $[0,1]$. The measure μ, which is an example of a *Gaussian multiplicative chaos measure*, turns out to be almost surely the zero measure if $\beta \geq \sqrt{2}$, but for $\beta < \sqrt{2}$ one gets a non-trivial limit. In the latter case μ is almost surely singular with respect to the Lebesgue measure, yet it has no atoms.

1 *A note on terminology*: In this introduction the word 'distribution' will always refer to continuous linear functionals on some space of test functions, typically tempered distributions à la Schwartz. Probability distributions will be called probability laws instead.

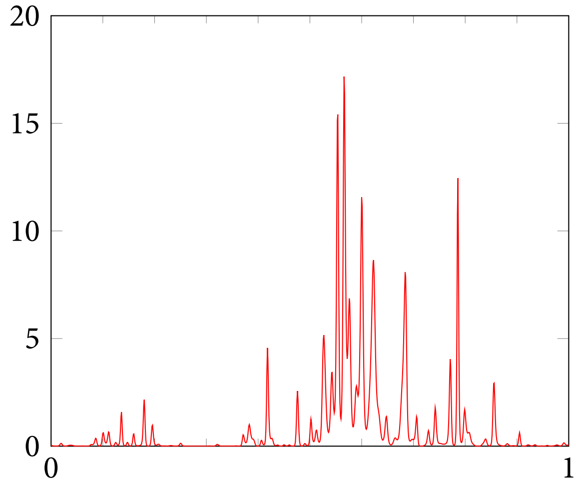


Figure 1.1. A computer simulation of $M_n(x)$ for $n = 1000$ and $\beta = 1$.

In Figure 1.1 we have plotted a single simulated realization of $M_n(x)$ when $n = 1000$ and $\beta = 1$. One can see that the function stays most of the time rather close to 0, but it has some large spikes. As n increases, the spikes will get thinner and taller, and in the limit the whole mass will actually be concentrated on some set of Hausdorff dimension strictly less than 1.

1.2 *History and current trends*

The history of multiplicative chaos traces back to the 1970s, where it appeared almost simultaneously and independently in two quite different contexts. First, in the 1971-paper [17] Raphael Høegh-Krohn considered certain quantum field theories whose Hamiltonians have the form $H = H_0 + V$, where V is a multiplicative chaos -type object. Second, in the 1972-article [24] Benoît Mandelbrot proposed a novel *limit log-normal* model for energy dissipation in turbulence.

Both Høegh-Krohn and Mandelbrot were able to study their models rigorously in the so called L^2-phase where the second moment of the multiplicative chaos is finite. Mandelbrot moreover provided heuristical arguments on how one might be able to go past this phase to

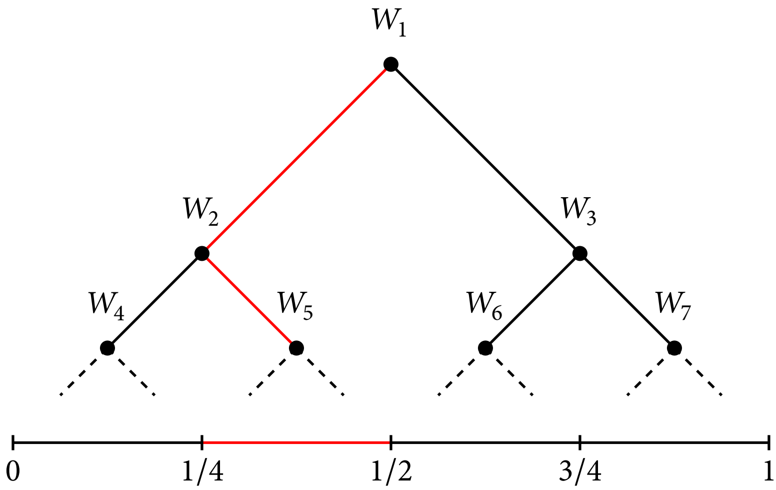


Figure 1.2. A dyadic Mandelbrot cascade on $[0, 1]$.

the whole subcritical phase.[2] Making these heuristics rigorous however turned out to be difficult, so he switched to study simplified models called *multiplicative cascades* in his two subsequent papers [25, 26]. Multiplicative cascades have also been called *Mandelbrot cascades*, and a basic example of a one-dimensional cascade built on the unit interval is illustrated in Figure 1.2. We will next explain the construction.

Briefly, the goal is to form a random measure on some space, say the unit interval as in the picture. We begin by dividing the interval recursively into left and right halves, starting from the interval $[0, 1]$ itself. Each dyadic subinterval can then be identified in a natural way with a node in the tree of splittings, which one can visualize to be hanging above the unit interval. Next we assign every node/interval k a non-negative random weight W_k. The nth level approximation μ_n for the cascade measure is then obtained by letting the measure of a dyadic subinterval I of length 2^{-n} be the product of the weight of I together with the weights of all the ancestors of I, distributing the mass uniformly inside I. In the picture I could be the red interval, and then the

2 In our introductory example the L^2-phase corresponds to having $\beta \in (0, 1)$, while the subcritical phase has $\beta \in (0, \sqrt{2})$.

corresponding weights would be the ones lying on the red path from the root of the tree to the node corresponding to I.

If we assume that the weights W_k are independent and identically distributed random variables with mean $1/2$, the sequence μ_n becomes an almost surely converging martingale with a limit measure μ. The precise condition for the non-degeneracy of the limit in this case is

$$\mathbb{E}\, W_1 \log(W_1) < 0\,,$$

a fact that was proven by Jean-Pierre Kahane and Jacques Peyrière in an article [20] that appeared two years after Mandelbrot's work. In the same paper the authors also proved many other important basic properties of μ, concerning e.g. the existence of moments and the support of the cascade measure.

The progress made on Mandelbrot cascades was thus relatively fast, but when the model is viewed as a simplification of the original limit log-normal model, it takes its toll on naturality. The problem is that the cascade measures would rather live on the boundary of a tree than in Euclidean space: The stochastic dependence between two locations x and y can vary drastically depending on where x and y are located in the space, even if their Euclidean distance is held constant. For instance, if $x = 1/2-\varepsilon$ and $y = 1/2+\varepsilon$ are points on the unit interval in Figure 1.2, then the lowest common ancestor of x and y in the tree is the root. This means that the behaviour of μ at x is almost completely independent of the behaviour at y, despite the points being very close to each other when $\varepsilon > 0$ is small.

This problem was overcome when Mandelbrot's original limit log-normal model was finally made rigorous by Kahane in 1985 when he published his seminal article [19]. In the paper did Kahane not only coin the term *multiplicative chaos* and present a solid mathematical theory capable of justifying and generalizing Mandelbrot's model, but he also proved many basic properties of the resulting chaos measures.

Kahane's theory allows one to construct multiplicative chaos measures in arbitrary locally compact metric spaces (T, ρ). Roughly speaking, Kahane showed how to define random measures of the form

$$e^{\beta X(x) - \frac{\beta^2}{2}\mathbb{E}\, X(x)^2}\, d\sigma\,, \tag{1.2}$$

where $\beta \in \mathbb{R}$ is a parameter, σ is a reference measure, and X is a Gaussian field on T. The field X however is typically not a function but a distribution, and it is not immediately clear how to make sense of (1.2). This will be further elaborated in Chapters 2 and 3.

In this introduction we will be focusing on the case where $T \subset \mathbb{R}^d$ and ρ is the usual Euclidean distance, in which case it turns out that the natural Gaussian fields to look at are those whose covariance has a logarithmic singularity on the diagonal, formally

$$\mathbb{E}\, X(x)X(y) = \log^+ \frac{1}{|x-y|} + O(1)\,.$$

This has also been the most important case for applications. Of the articles in this thesis, [I] is written for general metric spaces (with applications in $\mathbb{R}^d$), while [II] and [III] are written in the Euclidean setting.

It took some time before Kahane's theory started to receive more widespread interest, but today multiplicative chaos has connections to many directions. These include for example the Stochastic Loewner Evolution (SLE) [2, 15, 36], two-dimensional quantum gravity [9, 13, 14, 21], and number theory [16, 33]. Along with these rather recent developments, a call for the further study of multiplicative chaos distributions, including variants such as non-Gaussian and complex chaos, has emerged. The three papers included in this thesis provide some answers to this call:

In [I] we show that non-atomic real Gaussian multiplicative chaos measures are universal (especially in the so-called critical case), in the sense that various different ways of constructing the chaos actually yield the same result.

In [II] we construct non-Gaussian multiplicative chaos for fairly general log-correlated fields that admit a representation as a sum of independent fields.

Finally, in [III] we study the basic properties of purely imaginary Gaussian multiplicative chaos, which is perhaps the most elementary variant of complex multiplicative chaos.

This introduction discusses prerequisities and results from the literature that are related to the multiplicative chaos theory appearing in [I–III]. At the same time we will also state some selected theorems from the three papers.

2

LOG-CORRELATED GAUSSIAN FIELDS

2.1 *Distribution-valued Gaussian fields*

Let $U \subset \mathbb{R}^d$ be a bounded open set. Classically one would think of a Gaussian field on U as a random function $X: U \to \mathbb{R}$, such that for any finite collection of points $x_1, \dots, x_n \in U$ the random variables $X(x_1), \dots, X(x_n)$ are jointly Gaussian. In this chapter we will first shortly discuss how this concept can be generalized to more singular Gaussian fields on U, whose realizations are not functions but distributions. After this is done, we will define log-correlated Gaussian fields and provide some examples.

For simplicity, we will henceforth always assume that our Gaussian random variables are centered, meaning that they have expectation 0. The law of a Gaussian random vector $X \in \mathbb{R}^n$ is then completely determined by its covariance matrix C, which is an $n \times n$ positive semi-definite matrix with entries $C_{j,k} = \mathbb{E}\, X_j X_k$. From this it follows that the law of any Gaussian field X on U is determined by its *covariance function* $C_X(x, y) := \mathbb{E}\, X(x)X(y)$.[1] Conversely, given a positive definite function $C: U \times U \to \mathbb{R}$, one may construct (by using the Kolmogorov extension theorem) a collection $\{X(x) : x \in U\}$ of Gaussian random variables, in such a way that for any $n \geq 1$ and $x_1, \dots, x_n \in U$ the random variables $X(x_1), \dots, X(x_n)$ are jointly Gaussian with covariance matrix given by $(C(x_j, x_k))_{j,k=1}^n$.

This point of view, while natural, breaks down when we try to define log-correlated fields. Consider the example we had earlier in Chapter 1 where X is the Gaussian Fourier series given by (1.1). We will soon see that X is an example of a log-correlated field in the sense of the

1 Recall that for any index set I the law of any random variable $\Omega \to \mathbb{R}^I$ is determined by the laws of its finite-dimensional projections.

upcoming Definition 2.2 – for now, let us simply note that after a small (formal) computation one finds that

$$\mathbb{E}\, X(x)X(y) = \log \frac{1}{2|\sin(\pi(x-y))|}$$

for all $x, y \in [0,1]$. Notice that this would imply that $X(x)$ has infinite variance, so the idea that $X(x)$ is a Gaussian random variable is not going to work – X cannot be a random function.

It is however easy to check that X makes sense as a random distribution on the unit circle $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$. Indeed, if $\varphi \in C^\infty(\mathbb{T})$ is a test function, then its Fourier coefficients decay faster than any polynomial, while the coefficients $A_k/\sqrt{k}$ and $B_k/\sqrt{k}$ stay almost surely bounded. In fact, using the Borel–Cantelli lemma one easily sees that for any $\varepsilon > 0$ the random variables A_k and B_k are almost surely less than $\sqrt{(2+\varepsilon)\log(k)}$ for large enough k. It follows that X can be evaluated against any test function that has Fourier coefficients decaying faster than $(1+|k|)^{-1/2-\varepsilon}$ for some $\varepsilon > 0$.

The above example indicates that we should aim for a definition of a random Gaussian *distribution*. To this end, let $\mathcal{S}'$ be the space of tempered distributions on $\mathbb{R}^d$, where $\mathcal{S}$ denotes the Schwartz function space. We say that a real valued random distribution $X \in \mathcal{S}'$ is an $\mathcal{S}'$-*valued Gaussian field*, if the random variables $\{X(\varphi) : \varphi \in \mathcal{S}, \varphi \text{ is real}\}$ are jointly Gaussian. We have thus replaced the point evaluations in the earlier definition of a Gaussian random function by evaluations against test functions.

If X is an $\mathcal{S}'$-valued Gaussian field, we may define a bilinear form C_X on $\mathcal{S}$ by setting

$$C_X(f,g) := \mathbb{E}\, X(f)X(g).$$

This bilinear form is symmetric and positive definite, and it clearly determines the law of X. The converse is a bit trickier in this case. Given a symmetric and positive definite linear form C on $\mathcal{S}$, we can again find a collection of random variables $\{X(f) : f \in \mathcal{S}\}$ in such a way that the law of X agrees with C. However, it is not immediately clear that one can do this in such a way that the linear structure $X(f) + X(g) = X(f+g)$ is preserved, and more importantly, it is also not clear that one can choose the variables so that $X \in \mathcal{S}'$. Fortunately, the following simple corollary of Minlos theorem holds (see e.g. [37, Theorem 1.10] and the discussion following it).

Theorem 2.1. Let C be a real bilinear form on $\mathcal{S}$ that is symmetric, continuous and positive definite. Then there exists an $\mathcal{S}'$-valued Gaussian field X on $\mathbb{R}^d$ such that $C_X(f,g) = C(f,g)$ for all $f,g \in \mathcal{S}$.

2.2 *Log-correlated fields*

We are now ready to define log-correlated Gaussian fields.

Definition 2.2. Let $U \subset \mathbb{R}^d$ be a bounded open set. An $\mathcal{S}'$-valued Gaussian field X is *log-correlated* on U, if C_X is given by an integral

$$C_X(f,g) = \int_{\mathbb{R}^d \times \mathbb{R}^d} C_X(x,y) f(x) g(y)\, dx\, dy\,, \qquad (f,g \in \mathcal{S})$$

with kernel $C_X(x,y)$ of the form

$$C_X(x,y) = \begin{cases} \log \frac{1}{|x-y|} + g(x,y)\,, & \text{if } x,y \in U \\ 0\,, & \text{otherwise,} \end{cases}$$

where $g \in L^1(U \times U)$ is some integrable function that is bounded from below on compact subsets of U and bounded from above on all of U.

Remark. There is also an object known as *the* log-correlated Gaussian field (LGF) on $\mathbb{R}^d$. It has $g = 0$ and is defined on the whole space $\mathbb{R}^d$, albeit only up to an additive constant. See [12] for more details.

Remark. We would like to point out that the admittedly abstract Theorem 2.1 is not necessary for constructing log-correlated fields. One could for example consider the Karhunen–Loève expansion of the field and show that it converges in a suitable negative-index Sobolev space. Interested readers may wish to look at [**III**, Section 2], where this approach is carried out.

As already mentioned, (1.1) is an example of a log-correlated field. Let us mention another central example, which is the 2-dimensional Gaussian Free Field.

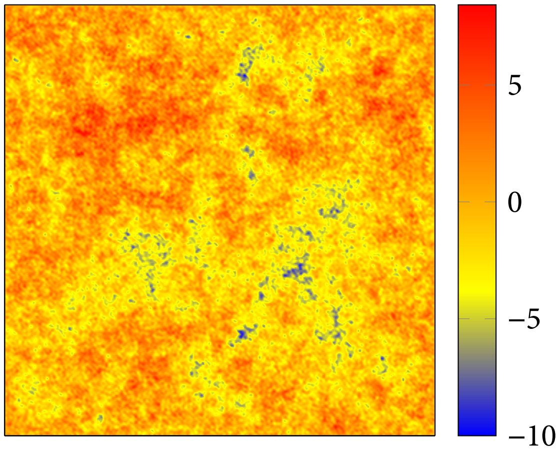


Figure 2.1. A computer simulation of the GFF in the unit square $[0, 1]^2$.

Definition 2.3. Let $U \subset \mathbb{C}$ be a simply connected bounded domain. The *Gaussian Free Field* (GFF) on U with zero boundary conditions is the log-correlated Gaussian field with the covariance kernel

$$C_X(x, y) = \log \left| \frac{1 - \varphi(x)\overline{\varphi(y)}}{\varphi(x) - \varphi(y)} \right| ,$$

where $\varphi : U \to \mathbb{D}$ is any conformal homeomorphism between U and the unit disc $\mathbb{D}$.

The GFF appears as the scaling limit of many models in mathematical physics, see [35] for an introduction to the topic. A central feature of the GFF is its domain Markov property, which states that the conditional law of a GFF X on some open subdomain $V \subset U$ given its values outside of V is equal to the sum of the harmonic extension of $X|_{\partial V}$ to V and an independent GFF in V.

When analyzing log-correlated Gaussian fields – and later Gaussian multiplicative chaos – some specific covariance kernels are particularly well-behaved. The *exactly scale invariant field* on the unit interval $[0, 1]$ has the pure logarithm as its covariance kernel:

$$C_X(x, y) = \log \frac{1}{|x - y|} . \tag{2.1}$$

This field has the property that if $0 < s < 1$ is a scaling parameter, then the law of $X(s\cdot)$ is the same as the law of X plus an indepedent Gaussian random variable with variance $\log(s^{-1})$.

More generally, the so called $\star$-*scale invariant* covariance kernels [1, 31] are the ones with a representation

$$C_X(x, y) = \int_1^\infty \frac{k((x-y)t)}{t}\, dt\,, \tag{2.2}$$

where k is a positive definite continuous function with $k(0) = 1$. If k is in addition (say) compactly supported, then these fields enjoy a useful spatial decorrelation property: The integrand in (2.2) is 0 for $t \gtrsim |x-y|^{-1}$.

2.3 *Approximations*

The definition of Gaussian multiplicative chaos in the next chapter relies on approximating log-correlated fields with functions. We will do this by using convolution approximations, since they work for any field X. However, certain other approximation paradigms also deserve to be mentioned.

For the Gaussian Fourier series (1.1) a natural approximating sequence of functions is given simply by the partial sums of the series (1.1). This method of approximation has independent increments, but its spatial decorrelation properties are poor since the trigonometric basis functions $\cos(2\pi kx)$ and $\sin(2\pi kx)$ are not localized. Such lack of spatial decorrelation is often an obstacle in proofs, because it makes it hard to use arguments that partition the space and claim that the behaviour of the field should be more or less independent in different parts.

Certain fields possess approximation schemes that feature both independent increments and spatial decorrelation. One particularly convenient one is the geometric construction of Emmanuel Bacry and Jean François Muzy [3], which is based on looking at cones of hyperbolic white noise in the upper half plane. Such representations exist – among other fields – both for (1.1) and (2.1). Details of the construction in the case of (1.1) can be found in [2].[2] In turn, for $\star$-scale invariant fields a good approximation is obtained simply by truncating the integral (2.2). We refer to [1] for more information.

For the GFF, a commonly used natural approximation is to take circle averages

$$X_\varepsilon(x) = \fint_{\partial B(x,\varepsilon)} X(t)\,dt$$

of the field. Due to the domain Markov property of the GFF these approximations possess spatial independence for distances larger than 2ε, and moreover for a fixed point x the process $\varepsilon \mapsto X_\varepsilon(x)$ has the covariance

$$\mathbb{E}\, X_\varepsilon(x) X_{\varepsilon'}(x) = \log \frac{1}{\max(\varepsilon, \varepsilon')} + \log \rho(x; U),$$

where $\rho(x;U)$ is the conformal radius of U as seen from x. This is essentially a time-scaled Brownian motion. Proofs and details can be found e.g. in [14].

Finally – just to mention yet another scheme – one can also approximate (1.1) using *vaguelets*, see e.g. [**I**, 38].

Remark. There is a substantial number of results that have only been proven for $\star$-scale invariant and similar well-approximable fields. This is rectified at least partially by [18, Theorem A], where the authors of [**III**] show that in fact any log-correlated Gaussian field with sufficiently regular covariance can be locally written as a sum of a $\star$-scale invariant field and a Hölder-regular field.

2 Strictly speaking the cone construction in [2] gives (1.1) *plus* an independent Gaussian random variable with variance $2\log(2)$.

3

GAUSSIAN MULTIPLICATIVE CHAOS

3.1 *Definitions*

Let $X \in \mathcal{S}'$ be a log-correlated Gaussian field on some bounded domain $U \subset \mathbb{R}^d$ as in Definition 2.2, and fix a parameter $\beta > 0$. A Gaussian multiplicative chaos (GMC) measure μ_β is formally constructed from X by taking a renormalized exponential:

$$d\mu_\beta(x) := e^{\beta X(x) - \frac{\beta^2}{2}\mathbb{E}\, X(x)^2}\, dx\,. \tag{3.1}$$

However, as we saw in Chapter 2, the field X is not a function, so (3.1) is not mathematically valid as it is. Therefore, to obtain a rigorous definition of μ_β, we will instead approximate X with regular fields for which a renormalized exponential can be defined, and then take the limit of such approximations.

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be a non-negative bump function with integral 1, and denote $\varphi_\varepsilon(x) := \varepsilon^{-d}\varphi(x/\varepsilon)$ for all $\varepsilon > 0$. It is easy to check that the functions φ_ε form an approximation of the identity for tempered distributions, in the sense that for any $h \in \mathcal{S}'$ and $f \in \mathcal{S}$ we have

$$\lim_{\varepsilon \to 0} \langle \varphi_\varepsilon * h, f\rangle = \langle h, f\rangle\,,$$

where $\varphi_\varepsilon * h$ is to be understood as the function $x \mapsto \langle h, \varphi_\varepsilon(x - \cdot)\rangle$.

Using φ, we may thus define the approximations

$$X_\varepsilon(x) := (\varphi_\varepsilon * X)(x)$$

for all $\varepsilon > 0$ and $x \in \mathbb{R}^d$. The functions X_ε will be almost surely smooth, and using them in place of X makes it possible to make sense of (3.1).

Definition 3.1. The GMC measure μ_β related to the log-correlated Gaussian field X is given by

$$d\mu_\beta(x) := \lim_{\varepsilon \to 0} e^{\beta X_\varepsilon(x) - \frac{\beta^2}{2}\mathbb{E}\, X_\varepsilon(x)^2}\, dx\,,$$

where the limit is in the sense of weak*-convergence in probability.

When $\beta \in (0, \sqrt{d})$ (the so called L^2-region) one can check that the definition makes sense and yields a non-trivial limit by showing that for any fixed $f \in C_c(U)$ the random variable

$$M_\varepsilon(f) := \int_U f(x) e^{\beta X_\varepsilon(x) - \frac{\beta^2}{2}\mathbb{E}\, X_\varepsilon(x)^2}\, dx$$

is Cauchy in $L^2(\Omega)$ as $\varepsilon \to 0$. After this is done, one can use the separability of $C_c(U)$ to show that there exists a limiting measure μ_β. Extending the result to all $\beta \in (0, \sqrt{2d})$ is however non-trivial.

Theorem 3.2. The limit in Definition 3.1 exists and is almost surely non-zero when $0 < \beta < \sqrt{2d}$. Moreover, the limit does not depend on the choice of φ.

A version of Theorem 3.2 was first proven by Kahane in [19], where instead of taking convolution approximations he considered fields that can be represented as a sum of independent fields with sufficiently regular positive covariances (the so called *σ-positivity condition*). The existence of chaos via convolution approximations was later proven by Raoul Robert and Vincent Vargas in the case that X is stationary [32]. Both in [19] and [32] the respective authors also proved the uniqueness of the resulting chaos for their respective approximation schemes: Two different σ-positive decompositions give the same result, as does using two different convolution kernels φ.

Today the approach taken by Nathanaël Berestycki in [7] is probably the most straightforward way to prove the existence and uniqueness of chaos when using convolution approximations, while the paper by Alexander Shamov [34] gives a robust novel definition of GMC along with very general existence and uniqueness results. Uniqueness can also be deduced from [**I**, Theorem 1], where the conditions are less general than in [34], but the result extends also to the critical setting $\beta = \sqrt{2d}$, which will be discussed in Chapter 4.

3.2 *Moments and support*

We will next list some properties of the chaos measure μ_β to give a feeling what kind of beast[1] we are talking about. Let us start with the moments of the total mass.

Theorem 3.3. Let $K \subset U$ be compact and assume that $0 < \beta < \sqrt{2d}$. Then $\mathbb{E}\,|\mu_\beta(K)|^p < \infty$ if and only if $p < \frac{2d}{\beta^2}$.

This theorem was proven by Kahane in [19] for $p \geq 0$, and for negative moments the result follows by mimicking the proof of corresponding theorem for multiplicative cascades [28], see also [32]. An often used strategy when proving results such as Theorem 3.3 is to show that the claim holds for some specific covariance kernels, after which it is possible to use the following fundamental inequality to extend the result to arbitrary kernels.

Theorem 3.4 (Kahane's convexity inequalities [19])**.** Let X and Y be Hölder-regular[2] Gaussian fields such that $C_X(x,y) \geq C_Y(x,y)$ for all $x, y \in U$. Then for any concave function $g\colon [0,\infty) \to [0,\infty)$ we have

$$\mathbb{E}\Big[g\Big(\int_U f(x) e^{X(x)-\frac{1}{2}\mathbb{E}\,X(x)^2}\,dx\Big)\Big] \leq \mathbb{E}\Big[g\Big(\int_U f(x) e^{Y(x)-\frac{1}{2}\mathbb{E}\,Y(x)^2}\,dy\Big)\Big]$$

for any non-negative $f \in C_c(U)$. For convex functions g with at most polynomial growth at infinity one gets the same inequality with sign reversed.

We have thus seen that the total mass is a rather heavy tailed random variable. The next theorem, also by Kahane apart from the exact Hausdorff dimension for which he only proved a lower bound, gives more precise information on the measure itself.

Theorem 3.5 ([19, 31])**.** The chaos measure μ_β is almost surely nonatomic. Moreover, it gives full mass to the set

$$\Big\{x \in U : \lim_{\varepsilon\to 0} \frac{X_\varepsilon(x)}{\mathbb{E}\,X_\varepsilon(x)^2} = \beta\Big\},$$

1 Nomenclature borrowed from Andriy Bondarenko.

2 By *Hölder-regular* we mean that the map $(x,y) \mapsto \sqrt{\mathbb{E}\,|X(x)-X(y)|^2}$ is Hölder-continuous.

which has Hausdorff dimension equal to $d - \frac{\beta^2}{2}$.

There are many further properties of the measures measures μ_β that have been investigated in the literature, for example the computation of its multifractal spectrum and asymptotics for the tail probabilities of the total mass. We refer the reader to the survey article [31] for more detailed information.

4

COMPLEX AND CRITICAL GAUSSIAN CHAOS

4.1 *Extending the range of β*

In Chapter 3 we considered Gaussian chaos μ_β for the parameter values $\beta \in (0, \sqrt{2d})$. It is trivial to extend this to the range $\beta \in (-\sqrt{2d}, \sqrt{2d})$, since for $\beta = 0$ the chaos is just the Lebesgue measure, and for $\beta < 0$ the measure μ_β has the same law as $\mu_{-\beta}$ by the symmetricity of X.

A natural question to ask is what happens when β is allowed to be complex. It turns out [2, 6] that at least for some specific fields X, such as the exactly scale invariant field (2.1), the range of subcritical β can be extended to the open region

$$\operatorname{int}(\operatorname{conv}(\{z \in \mathbb{C} : |z| = \sqrt{d}\} \cup \{-\sqrt{2d}, \sqrt{2d}\}))$$

which is illustrated in Figure 4.1.[1] More precisely, the standard martingale normalization yields a non-trivial limit for β lying in this eye-shaped domain. The disc in the middle corresponds to the L^2-phase.

4.2 *Purely imaginary chaos*

One particularly interesting region in Figure 4.1 is the imaginary axis. For notational simplicity, we will keep β real in this section and instead consider the parameter $i\beta$. The distribution $\mu_{i\beta}$ for $0 < \beta < \sqrt{d}$ is then formally given by

$$\mu_{i\beta} = e^{i\beta X(x) + \frac{\beta^2}{2}\mathbb{E}\, X(x)^2}, \tag{4.1}$$

which is again to be rigorously understood via a regularization procedure. The study of $\mu_{i\beta}$ is the main topic of [**III**]. There is a plot of a computer simulation of the real part of an approximation of $\mu_{i\beta}$ in Figure 4.2. In the simulation the underlying field X is the GFF in the unit

1 Here for $A \subset \mathbb{C}$ we denote by $\operatorname{int}(A)$ and $\operatorname{conv}(A)$ the interior and convex hull of A, respectively.

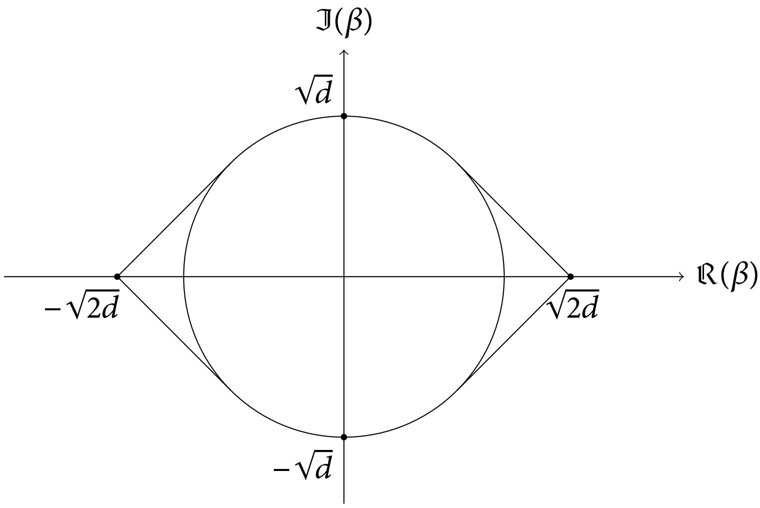


Figure 4.1. The extended subcritical regime for complex β.

square – actually the same realization as in Figure 2.1. The parameter value is $\beta = 1/\sqrt{2}$.

Notice how the role played by the normalization in (4.1) is quite different from what it was in the case of the real chaos: There we had to apply a normalizing factor that tends to 0 in order to counter the more and more probable very large values of X_ε. In the imaginary case we instead have to renormalize by a factor that blows up, so that the ever more wildly oscillating term $\exp(i\beta X_\varepsilon(x))$ does not bring the limit to 0.

A central feature of the purely imaginary chaos distributions is that they possess all moments, a fact that is not true for other parameter values.

Theorem 4.1 ([III, Theorem 1.3])**.** For any $f \in C_c(U)$ we have

$$\mathbb{E}\,|\mu_{i\beta}(f)|^p < \infty$$

for all $p \geq 1$. Moreover, the law of $\mu_{i\beta}$ is determined by its moments.

The purely imaginary chaos is an honest distribution and not even a complex measure.

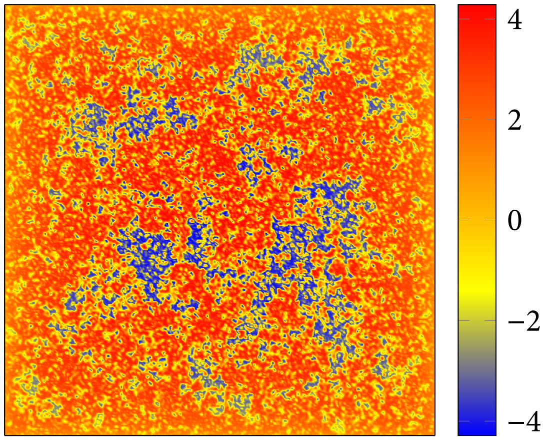


Figure 4.2. A computer simulation of the real part of the imaginary chaos of the GFF in the unit square $[0,1]^2$.

Theorem 4.2 ([III, Theorem 1.2])**.** The distribution $\mu_{i\beta}$ has infinite total variation and is hence almost surely not a complex measure. It belongs almost surely to the Besov space $B^s_{p,q}(\mathbb{R}^d)$ when $s < -\beta^2/2$, and this bound is sharp except possibly at $s = -\beta^2/2$. In particular it belongs to the L^2-Sobolev space $H^s(\mathbb{R}^d)$ for $s < -\beta^2/2$.

Yet another interesting feature of the imaginary chaos is that when suitably renormalized it becomes white noise as $\beta \to \sqrt{d}$.

Theorem 4.3 ([III, Theorem 3.20])**.** As $\beta \to \sqrt{d}$, we have $\sqrt{\frac{d-\beta^2}{|S^{d-1}|}}\mu_{i\beta} \to e^{\frac{\beta^2}{2}g(x,x)}W$ in law, where W is the standard complex white noise on U and g is the one appearing in Definition 2.2.

The purely imaginary measure emerges in the scaling limit of various models in mathematical physics. One of the models we discuss in [III] is the so called XOR-Ising model. This model consists of two independent copies of the Ising model with spins multiplied together. We show that the scaling limit of the spin field of the critical XOR-Ising model converges in law to the real part of purely imaginary chaos $\mu_{i\beta}$ constructed from the GFF on the domain with parameter $\beta = 1/\sqrt{2}$. Figure 4.2 corresponds to this situation.

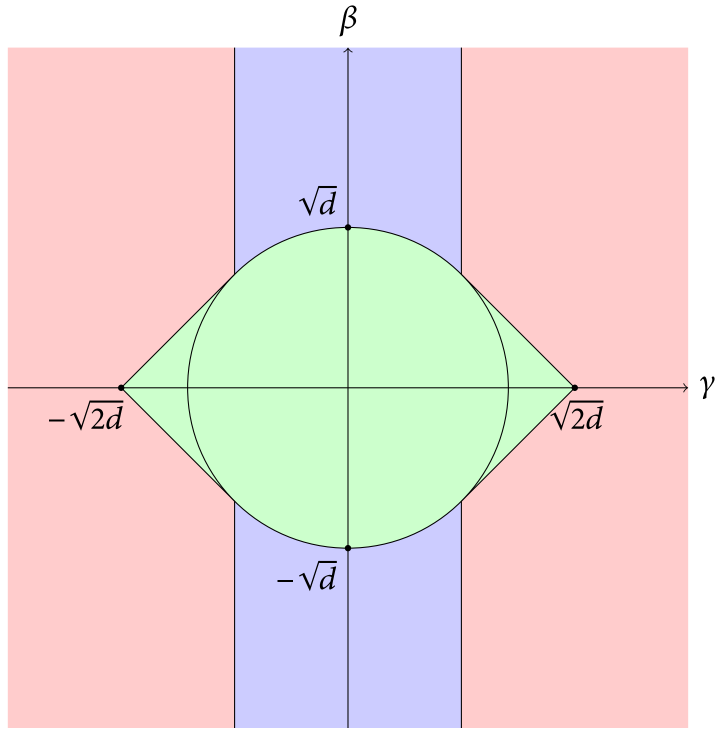


Figure 4.3. The phase diagram of [22].

4.3 *Other types of complex chaos*

Let us also briefly mention that choosing β to be complex is just one way to obtain complex versions of GMC. In [22] the authors study a version where the field is of the form $\gamma X(x) + i\beta Y(x)$, where X and Y are independent log-correlated Gaussian fields, and γ and β are two real parameters. In this situation they obtain a phase diagram as in Figure 4.3. They call the green part phase I, the red part phase II, and the blue part phase III.

Phase I corresponds to the subcritical regime as in Figure 4.1, and here one builds the chaos using the standard way of approximation

$$e^{\gamma X_\varepsilon(x)+i\beta Y_\varepsilon(x)-\frac{\gamma^2}{2}\mathbb{E}\,X_\varepsilon(x)^2+\frac{\beta^2}{2}\mathbb{E}\,Y_\varepsilon(x)^2},$$

which is normalized in such a way that the mean is 1. The same holds at the boundary between phase I and phase II (excluding the critical points $\gamma = \pm\sqrt{2d}$, $\beta = 0$ and the triple points $\gamma = \beta = \pm\sqrt{d/2}$), but at other locations one has to introduce additional normalization factors. The limits outside of phase I or the boundary between phases I and II (excluding critical and triple points) are complex white noise measures with control measures based on real multiplicative chaos, see [22] for details.

Yet another version of complex chaos appears in [33]. This is something one could call *analytic* or *Hardy* chaos, since it can be seen as the boundary values of a random analytic function. In [33] this analytic function arises from random statistics of the Riemann ζ-function on the critical line.

4.4 *Critical chaos*

As mentioned in the previous section, the so called critical chaos corresponds to the situation $\beta = \sqrt{2d}$ (we are back to the usual normalization with β denoting the real parameter). There are two approaches to obtaining a non-trivial measure μ_β in this case: The first one is the so called *derivative martingale* approach, where one looks at

$$\begin{aligned} D_\varepsilon(x) &:= -\frac{\partial}{\partial\beta}\Big[e^{\beta X_\varepsilon(x) - \frac{\beta^2}{2}\mathbb{E}\,X_\varepsilon(x)^2}\Big]_{\beta=\sqrt{2d}} \\ &= (\sqrt{2d}\mathbb{E}\,X_\varepsilon(x)^2 - X_\varepsilon(x))e^{\sqrt{2d}X_\varepsilon(x) - d\mathbb{E}\,X_\varepsilon(x)^2}\,. \end{aligned}$$

The second approach is to use the *Seneta–Heyde normalization*

$$M_\varepsilon(x) := \sqrt{\frac{\pi}{2}}\sqrt{\mathbb{E}\,X_\varepsilon(x)^2}\,e^{\sqrt{2d}X_\varepsilon(x) - d\mathbb{E}\,X_\varepsilon(x)^2}\,,$$

where we have introduced an additional renormalizing factor which grows like the square root of the variance of the approximation.

The convergence of $D_\varepsilon(x)\,dx$ to a non-trivial non-atomic measure $\mu_{\sqrt{2d}}$ was proven in [10] for the natural martingale approximation of $\star$-scale invariant fields, while in the subsequent paper [11] the same authors showed that M_ε converges to the same measure. Further properties such as the exact asymptotics for the tail of the distribution of the total mass were proven in [4].

In [I] we show the convergence of the Seneta–Heyde normalization for a large class of approximation schemes, provided that we a priori know the convergence for some approximation to which we can compare. The main tool in this is the following theorem.

Theorem 4.4 ([I, Theorem 1.1])**.** Let $(X_n)_{n=1}^\infty$ and $(\widetilde{X}_n)_{n=1}^\infty$ be two sequences of Hölder-regular Gaussian fields on a compact doubling metric space (T, d), with covariance functions $C_n(x, y)$ and $\widetilde{C}_n(x, y)$, respectively. Let ρ_n be a sequence of non-negative Radon reference measures on T. Define the sequence of measures

$$d\mu_n(x) \coloneqq e^{X_n(x) - \frac{1}{2}\mathbb{E}\, X_n(x)^2}\, d\rho_n(x)$$

and similarly define $\widetilde{\mu}_n$ by using the fields $\widetilde{X}_n$ instead. Assume that $\widetilde{\mu}_n$ converges in law to an almost surely non-atomic random measure $\widetilde{\mu}$. Suppose that the covariances C_n and $\widetilde{C}_n$ satisfy the following two conditions: There exists a finite constant $K > 0$ such that

$$\sup_{x,y \in T} |C_n(x, y) - \widetilde{C}_n(x, y)| \leq K \quad \text{for all } n \geq 1\,,$$

and

$$\lim_{n\to\infty} \sup_{d(x,y)>\delta} |C_n(x, y) - \widetilde{C}_n(x, y)| = 0 \quad \text{for all } \delta > 0\,.$$

Then also the measures μ_n converge in distribution to the same measure $\widetilde{\mu}$.

The role of the measures ρ_n in Theorem 4.4 is to allow for arbitrary deterministic normalizations, and in the case of Seneta-Heyde normalization one can simply choose

$$d\rho_n(x) = \sqrt{\frac{\pi}{2}} \sqrt{\mathbb{E}\, X_{\varepsilon_n}(x)^2}\, dx\,,$$

where $(\varepsilon_n)_{n=1}^\infty$ is some sequence tending to 0 from above.

There are still a number of situations where uniqueness results are not known, especially in the complex or non-Gaussian setting. For the real critical chaos of $\star$-scale invariant fields, the recent article by Ellen Powell [29] extends the uniqueness to the derivative normalization setting in the case of convolution approximations.

5

NON-GAUSSIAN CHAOS

5.1 *History and applications*

This final chapter concerns multiplicative chaos in a non-Gaussian setting. For multiplicative cascades the non-Gaussian situation was studied already by Kahane and Peyrière in [20], but for log-correlated non-Gaussian random fields in $\mathbb{R}^d$ the theory is much less understood. In the latter case the research has mainly been focused on infinitely divisible fields [3, 30] and on random multiplicative pulses [5, 6].

Non-Gaussian chaos appears naturally in various applications, since in many cases the model itself is not log-normal, even if it might in the scaling limit converge to GMC. For instance, in [**III**] we show that the scaling limit of the XOR-Ising model is the real part of a purely imaginary chaos distribution, but the XOR-Ising model itself is not log-normal. Another example is given in [33], where the model comes from the statistic behaviour of the Riemann ζ-function on the critical line, and in the limit one gets a certain GMC-type object *times* a smooth non-log-normal part. Further examples appear in the study of characteristic polynomials of random matrices [8, 23, 39].

In the above applications the approximations of the GMC are not log-normal, but the limit itself is still a GMC-distribution (perhaps with some additional smooth random factor). In [**II**] we look at a more general situation, where the resulting chaos itself might not be limit log-normal. A basic example is given by constructing chaos once again using the random Fourier series (1.1) from Chapter 1, but replacing the Gaussian random variables A_k and B_k by non-Gaussian ones. This and another example related to a construction of Petteri Mannersalo, Ilkka Norros, and Rudolf Riedi [27] are discussed in [**II**] as applications of a more general theorem for convergence of non-Gaussian chaos.

5.2 *Our approach*

In [II] we take a martingale approach to non-Gaussian multiplicative chaos, a bit like in the original work [19] by Kahane for the GMC. This way we do not have to care about the field itself, just its approximations. Our proof of existence of non-trivial chaos in this setting is on the other hand inspired by Berestycki's proof in [7].

Our starting point is a sequence $(X_k)_{k\geq 1}$ of real-valued, continuous, independent, and centered random fields on (let's say) the unit cube $I := [0,1]^d \subset \mathbb{R}^d$. We assume that this sequence is log-correlated in the following sense.

Definition 5.1. The sequence $(X_k)_{k\geq 1}$ has a *locally log-correlated structure* if the following conditions hold:

- $\sup_{x\in I} \mathbb{E}\, X_k(x)^2 \to 0$ and $\sum_{k=1}^{\infty} \mathbb{E}\, X_k(0)^2 = \infty$.
- There exists a constant $\delta > 0$ such that for all $n \geq 1$ and $x, y \in I$ with $|x-y| \leq \delta$ we have
$$\Big|\sum_{k=1}^{n} \mathbb{E}\, X_k(x)X_k(y) - \min\Big(\log\frac{1}{|x-y|}, \sum_{k=1}^{n} \mathbb{E}\, X_k(0)^2\Big)\Big| \leq C$$
for some constant $C > 0$.

The above definition is motivated by the Gaussian case, where the second point appears in the definition of the so called *standard approximation sequence* [III, Definition 2.7]. As in the Gaussian case, our goal is to show that the sequence of distributions

$$\mu_n(x) := \frac{e^{\beta \sum_{k=1}^n X_k(x)}}{\mathbb{E}\, e^{\beta \sum_{k=1}^n X_k(x)}}$$

has a non-trivial limit when $\beta \in (0, \sqrt{2d})$. In order to prove this, we need to make some additional regularity assumptions on the fields. The first of these conditions ensures that for *single points* $x \in I$ the sum $\sum_{k=1}^n X_k(x)$ will obey the central limit theorem as $n \to \infty$, so that it starts to appear Gaussian in a quantifiable way.

$$\sup_{x\in I} \sum_{k=1}^{\infty} \big(\mathbb{E}\, |X_k(x)|^{3+\varepsilon}\big)^{\frac{3}{3+\varepsilon}} < \infty \quad \text{for some } \varepsilon > 0\,. \tag{5.1}$$

The second condition is used in the proof for a large-deviations estimate on the supremum of the field.

$$\mathbb{E}\left|\sum_{k=1}^{n}(X_k(x)-X_k(y))\right|^r \le C_r e^{r\sum_{k=1}^{n}\mathbb{E}\,X_k(0)^2}|x-y|^r \text{ for } n,r\ge 1\,. \quad (5.2)$$

Finally, the fields X_k should have pointwise exponential moments,

$$\sup_{x\in I}\sup_{k\ge 1} e^{\lambda X_k(x)} < \infty \quad \text{for all } \lambda\in\mathbb{R}\,. \quad (5.3)$$

The above conditions hold especially for the random Fourier series (1.1), when instead of Gaussianity one simply assumes that the variables A_k and B_k are i.i.d. and satisfy $\mathbb{E}\,e^{\lambda A_1}<\infty$ for all $\lambda\in\mathbb{R}$.

The main result of [II] may now be stated as follows.

Theorem 5.2. Assume that $(X_k)_{k\ge1}$ is locally log-correlated as in Definition 5.1 and satisfies (5.1), (5.2), and (5.3). Then there exists an open $U\subset\mathbb{C}$ with $(0,\sqrt{2d})\subset U$ such that for any compact $K\subset U$ there exists $p=p_K>1$ for which the martingale $\mu_n(f)$ converges in $L^p(\Omega)$ to a limit $\mu(f;\beta)$ for all $\beta\in K$ and continuous $f: K\to\mathbb{C}$.

In [II] we also show that the convergence takes place in a suitable Sobolev space, and that for a fixed f the map $\beta\mapsto\mu(f;\beta)$ is almost surely analytic with respect to β. Moreover, the interval $(0,\sqrt{2d})$ is essentially optimal as in the Gaussian case, in the sense that if $\beta>\sqrt{2d}$ then the resulting measure is almost surely zero.

5.3 *Proving convergence*

Let us close this final chapter with a more detailed discussion on the proof of Theorem 5.2 in [II]. The same proof of course works also for standard GMC. Like the proof of Berestycki [7], it is based on separately looking at those points where the field is large and those where it is small. However, our refined method also works for complex β and it directly yields L^p-integrability. A small disclaimer is in place, though: The region of convergence in the complex plane is not optimal apart from what happens on the real axis, and the p one could extract from the proof is not optimal either.

The proof is based on partitioning the points $x \in I$ into classes based on the last level on which the field is large at x. Assume for simplicity that $\mathbb{E}\, X_k(x)^2 = 1$ for all $k \geq 1$ and $x \in I$, and define $Y_n(x) = \sum_{k=1}^n X_k(x)$. We say that the field is large on level l at the point x, if $Y_l(x) \geq \alpha \mathbb{E}\, Y_l(x)^2$, where $\alpha > \beta$ is some fixed constant that is chosen during the proof. Thus if n is some large natural number, we say that a point $x \in I$ belongs to the level $l \leq n$, if $Y_l(x) \geq \alpha \mathbb{E}\, Y_l(x)^2$ and $Y_k(x) < \alpha \mathbb{E}\, Y_k(x)^2$ for $l+1 \leq k \leq n$.

The event $Y_l(x) \geq \alpha \mathbb{E}\, Y_l(x)^2$ should be compared with Theorem 3.5. Since $\alpha > \beta$, the points for which this happens infinitely often do not contribute to the limit. This is because the probability of the event happening is very small for large l. On the other hand, assuming that $Y_k(x) < \alpha \mathbb{E}\, Y_k(x)^2$ for large enough k is enough to remove the extreme behaviour that makes the L^2-norm of $\mu_n(x)$ blow up, thus opening the door to L^2-arguments.

Assume that we wish to show that $\sup_{n \geq 1} \mathbb{E}\, |\mu_n(I)|^p < \infty$. Roughly speaking, the idea is to handle the contribution of the points belonging to a fixed level l by dividing I into dyadic intervals of length 2^{-l}. On such an interval J the field $Y_l(x)$ will not vary too much, and indeed it turns out that for small enough $p > 1$ we have

$$\mathbb{E} \sup_{x \in J} |\mu_l(x)|^p \mathbb{1}_{\{Y_l(x) \geq \alpha \mathbb{E}\, Y_l(x)^2\}} \lesssim e^{-\varepsilon l} \tag{5.4}$$

for some $\varepsilon > 0$. The right hand side is summable, so we would be done if we could somehow bound the contribution coming from the $Y_n(x) - Y_l(x)$ part of the field. This can be done by computing the conditional second moment of

$$\int_J \mu_n(x) \mathbb{1}_{\{Y_l(x) \geq \alpha \mathbb{E}\, Y_l(x)^2\}} \mathbb{1}_{\{Y_k(x) < \alpha \mathbb{E}\, Y_k(x)^2 \text{ for all } l+1 \leq k \leq n\}} \, dx$$

with respect to the σ-algebra $\mathcal{F}_l$ generated by $X_1, \dots, X_l$ and showing that it is bounded from above by

$$2^{-2ld} \sup_{x \in J} |\mu_l(x)|^2 . \tag{5.5}$$

5.4 *Open questions*

Many basic questions are open for our model of non-Gaussian chaos. For example, we do not know whether for complex β the subcritical phase is again the one illustrated in Figure 4.1, as it is in the Gaussian case. Interesting would also be to prove the convergence at criticality (using for example the Seneta–Heyde normalization).

Another topic related to convergence is universality. As discussed earlier in this introduction, there exist various results in the Gaussian case showing that one obtains the same chaos when using different approximations of the log-correlated field. Similar results in the non-Gaussian case are missing – ideally we would like to have robust theorems that do not require the martingale structure to show convergence and that also establish the uniqueness of the resulting chaos in some sense.

Finally, it would be interesting to look at the finer properties of non-Gaussian chaos distributions. Such properties include the optimal L^p-integrability, Sobolev regularity, and asymptotics for the tail probabilities of the total mass of the chaos, as well as the multifractal spectrum of the chaos measure on the real line.

BIBLIOGRAPHY

[1] R. Allez, R. Rhodes, and V. Vargas. *Lognormal $\star$-scale invariant random measures.* Probability Theory and Related Fields 155.3–4 (2013), 751–788.

[2] K. Astala, A. Kupiainen, E. Saksman, and P. Jones. *Random conformal weldings.* Acta mathematica 207.2 (2011), 203–254.

[3] E. Bacry and J. F. Muzy. *Log-infinitely divisible multifractal processes.* Communications in Mathematical Physics 236.3 (2003), 449–475.

[4] J. Barral, A. Kupiainen, M. Nikula, E. Saksman, and C. Webb. *Basic properties of critical lognormal multiplicative chaos.* The Annals of Probability 43.5 (2015), 2205–2249.

[5] J. Barral and B. Mandelbrot. *Multifractal products of cylindrical pulses.* Probability Theory and Related Fields 124.3 (2002), 409–430.

[6] J. Barral and B. Mandelbrot. *Random multiplicative multifractal measures.* Fractal Geometry and Applications: A Jubilee of Benoît Mandelbrot. Proceedings of Symposia in Pure Mathematics. Providence, Rhode Island: American Mathematical Society, 2004.

[7] N. Berestycki. *An elementary approach to Gaussian multiplicative chaos.* Electronic Communications in Probability 22 (2017).

[8] N. Berestycki, C. Webb, and M. D. Wong. *Random Hermitian matrices and Gaussian multiplicative chaos.* Probability Theory and Related Fields (2017), 1–87.

[9] F. David, A. Kupiainen, R. Rhodes, and V. Vargas. *Liouville quantum gravity on the Riemann sphere.* Communications in Mathematical Physics 342.3 (2016), 869–907.

[10] B. Duplantier, R. Rhodes, S. Sheffield, and V. Vargas. *Critical Gaussian multiplicative chaos: convergence of the derivative martingale*. The Annals of Probability 42.5 (2014), 1769–1808.

[11] B. Duplantier, R. Rhodes, S. Sheffield, and V. Vargas. *Renormalization of Critical Gaussian Multiplicative Chaos and KPZ Relation*. Communications in Mathematical Physics 330.1 (2014), 283–330.

[12] B. Duplantier, R. Rhodes, S. Sheffield, and V. Vargas. *Log-correlated Gaussian fields: an overview*. Geometry, Analysis and Probability. Cham: Birkhäuser, 2017, 191–216.

[13] B. Duplantier and S. Sheffield. *Duality and the Knizhnik–Polyakov–Zamolodchikov relation in Liouville quantum gravity*. Physical Review Letters 102.15 (2009), 150603.

[14] B. Duplantier and S. Sheffield. *Liouville quantum gravity and KPZ*. Inventiones mathematicae 185.2 (2011), 333–393.

[15] B. Duplantier and S. Sheffield. *Schramm-Loewner Evolution and Liouville Quantum Gravity*. Physical Review Letters 107.13 (2011), 131–305.

[16] A. J. Harper. *Moments of random multiplicative functions, I: Low moments, better than squareroot cancellation, and critical multiplicative chaos*. arXiv:1703.06654 (2017).

[17] R. Høegh-Krohn. *A general class of quantum fields without cut-offs in two space-time dimensions*. Communications in Mathematical Physics 21.3 (1971), 244–255.

[18] J. Junnila, E. Saksman, and C. Webb. *Decompositions of log-correlated fields with applications*. arXiv:1808.06838 (2018).

[19] J.–P. Kahane. *Sur le chaos multiplicatif*. Comptes rendus de l'Académie des sciences. Série 1, Mathématique 301.6 (1985), 329–332.

[20] J.–P. Kahane and J. Peyrière. *Sur certaines martingales de Benoit Mandelbrot*. Advances in Mathematics 22.2 (1976), 131–145.

[21] A. Kupiainen, R. Rhodes, and V. Vargas. *Integrability of Liouville theory: proof of the DOZZ Formula*. arXiv:1707.08785 (2017).

[22] H. Lacoin, R. Rhodes, and V. Vargas. *Complex Gaussian Multiplicative Chaos*. Communications in Mathematical Physics 337.2 (2015), 569–632.

[23] G. Lambert, D. Ostrovsky, and N. Simm. *Subcritical multiplicative chaos for regularized counting statistics from random matrix theory*. Communications in Mathematical Physics 360.1 (2018), 1–54.

[24] B. Mandelbrot. *Possible refinement of the lognormal hypothesis concerning the distribution of energy dissipation in intermittent turbulence*. Statistical Models and Turbulence. Berlin, Heidelberg: Springer, 1972, 333–351.

[25] B. Mandelbrot. *Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier*. Journal of Fluid Mechanics 62.2 (1974), 331–358.

[26] B. Mandelbrot. *Multiplications aléatoires itérées et distributions invariantes par moyenne pondérée aléatoire*. Comptes Rendus Mathématique. Académie des Sciences. Paris 278 (1974), 289–292, 355–358.

[27] P. Mannersalo, I. Norros, and R. H. Riedi. *Multifractal products of stochastic processes: construction and some basic properties*. Advances in Applied Probability 34.4 (2002), 888–903.

[28] G. M. Molchan. *Scaling exponents and multifractal dimensions for independent random cascades*. Communications in Mathematical Physics 179.3 (1996), 681–702.

[29] E. Powell. *Critical Gaussian chaos: convergence and uniqueness in the derivative normalisation*. Electronic Journal of Probability 23 (2018).

[30] R. Rhodes, J. Sohier, and V. Vargas. *Levy multiplicative chaos and star scale invariant random measures*. The Annals of Probability 42.2 (2014), 689–724.

[31] R. Rhodes and V. Vargas. *Gaussian multiplicative chaos and applications: a review*. Probability Surveys 11 (2014), 315–392.

[32] R. Robert and V. Vargas. *Gaussian multiplicative chaos revisited*. The Annals of Probability 38.2 (2010), 605–631.

[33] E. Saksman and C. Webb. *The Riemann zeta function and Gaussian multiplicative chaos: statistics on the critical line*. arXiv:1609.00027 (2016).

[34] A. Shamov. *On Gaussian multiplicative chaos*. Journal of Functional Analysis 270.9 (2016), 3224–3261.

[35] S. Sheffield. *Gaussian free fields for mathematicians*. Probability Theory and Related Fields 139.3–4 (2007), 521–541.

[36] S. Sheffield. *Conformal weldings of random surfaces: SLE and the quantum gravity zipper*. The Annals of Probability 44.5 (2016), 3474–3545.

[37] B. Simon. *The $P(\Phi)_2$ Euclidean (Quantum) Field Theory*. Princeton, New Jersey: Princeton University Press, 2015.

[38] N. Tecu. *Random conformal weldings at criticality*. arXiv:1205.3189 (2012).

[39] C. Webb. *The characteristic polynomial of a random unitary matrix and Gaussian multiplicative chaos – The L^2-phase*. Electronic Journal of Probability 20 (2015).

ARTICLE I

J. Junnila and E. Saksman

Uniqueness of critical Gaussian chaos

Electronic Journal of Probability 22 (*2017*)

Electronic Journal of Probability

Electron. J. Probab. **22** (2017), no. 11, 1–31.
ISSN: 1083-6489 DOI: 10.1214/17-EJP28

Uniqueness of critical Gaussian chaos*

Janne Junnila† Eero Saksman‡

Abstract

We consider Gaussian multiplicative chaos measures defined in a general setting of metric measure spaces. Uniqueness results are obtained, verifying that different sequences of approximating Gaussian fields lead to the same chaos measure. Specialized to Euclidean spaces, our setup covers both the subcritical chaos and the critical chaos, actually extending to all non-atomic Gaussian chaos measures.

Keywords: multiplicative chaos; uniqueness; critical temperature.
AMS MSC 2010: Primary 60G57; 60G15, Secondary 60K35; 60G60.
Submitted to EJP on April 28, 2016, final version accepted on January 23, 2017.
Supersedes arXiv:1506.05099.

1 Introduction

The theory of multiplicative chaos was created by Kahane [20, 21] in the 1980's in order to obtain a continuous counterpart of the multiplicative cascades, which were proposed by Mandelbrot in early 1970's as a model for turbulence. During the last 10 years there has been a new wave of interest on multiplicative chaos, due to e.g. its important connections to Stochastic Loewner Evolution [3, 29, 15], quantum field theories and quantum gravity [18, 13, 14, 24, 6, 23], models in finance and turbulence [25, Section 5], and the statistical behaviour of the Riemann zeta function over the critical line [16, 27].

In Kahane's original theory one considers a sequence of a.s. continuous and centered Gaussian fields X_n that can be thought of as approximations of a (possibly distribution valued) Gaussian field X. The fields are defined on some metric measure space $(\mathcal{T}, \lambda)$

*JJ was supported by the Doctoral Programme in Mathematics and Statistics at University of Helsinki. ES was supported by The Finnish Centre of Excellence (CoE) in Analysis and Dynamics Research.

†University of Helsinki, Department of Mathematics and Statistics, P.O. Box 68, FIN-00014 University of Helsinki, Finland
E-mail: janne.junnila@helsinki.fi

‡University of Helsinki, Department of Mathematics and Statistics, P.O. Box 68, FIN-00014 University of Helsinki, Finland
E-mail: eero.saksman@helsinki.fi

and the increments $X_{n+1} - X_n$ are assumed to be independent. One may then define the random measures μ_n on $\mathcal{T}$ by setting

$$\mu_n(dx) := \exp(X_n(x) - \frac{1}{2}\mathbb{E}\, X_n(x)^2)\lambda(dx).$$

In this situation basic martingale theory verifies that almost surely there exists a (random) limit measure $\mu = \lim_{n\to\infty} \mu_n$, where the convergence is understood in the weak^*-sense. The measure μ is called the *multiplicative chaos* defined by X (or rather by the sequence (X_n)), and Kahane shows that the limit does not depend on the choice of the approximating sequence (X_n), assuming that the covariances of the increments $X_{n+1} - X_n$ are non-negative. However, the limit may well reduce to the zero measure almost surely.

We next recall some of the most important cases of multiplicative chaos in the basic setting where $\mathcal{T}$ is a subset of a Euclidean space, say $\mathcal{T} = [0,1]^d$, and λ is the Lebesgue measure. Especially we assume that the limit field X is log-correlated, i.e. it has the covariance

$$C_X(x,y) = 2d\beta^2 \log|x-y| + G(x,y), \qquad x, y \in \mathcal{T}, \tag{1.1}$$

where G is a continuous and bounded function. As an important example in dimension 2, the Gaussian free field has locally such a covariance structure.

Assuming that the X_n are nice approximations of the field X as explained above, Kahane's theory yields that in case $\beta \in (0,1)$ the convergence $\mu_n \overset{w^*}{\to} \mu_\beta$ takes place almost surely and the obtained chaos μ_β is non-trivial. It is an example of *subcritical Gaussian chaos*, and, as we shall soon recall in more detail, in this normalisation $\beta = 1$ appears as a critical value.

In order to give a more concrete view of the chaos we take a closer look at a particularly important example of approximating Gaussian fields in the case where $d = 1$ and μ is the so-called exactly scale invariant chaos due to Bacry and Muzy [4], [25, p. 331]. Consider the hyperbolic white noise W in the upper half plane $\mathbb{R}^2_+$ so that $\mathbb{E}\, W(A_1)W(A_2) = m_{\mathrm{hyp}}(A_1 \cap A_2)$ for Borel subsets $A_1, A_2 \in \mathbb{R}^2_+$ with compact closure in $\mathbb{R}^2_+$. Above $dm_{\mathrm{hyp}} = y^{-2}dx\,dy$ denotes the hyperbolic measure in the upper half plane. For every $t > 0$ consider the set

$$A_t(x) := \{(x', y') \in \mathbb{R}^2_+ : y' \geq \max(e^{-t}, 2|x'-x|) \text{ and } |x'-x| \leq 1/2\} \tag{1.2}$$

and define the field X_t on $[0,1]$ by setting

$$X_t(x) := \sqrt{2d}W(A_t(x)).$$

Note that the sets $A_t(x)$ are horizontal translations of the set $A_t(0)$. One then defines the subcritical exactly scale invariant chaos by setting

$$d\mu_\beta(x) \overset{\text{a.s}}{:=} \lim_{t\to\infty} \exp\left(\beta X_t(x) - \frac{\beta^2}{2}\mathbb{E}\,(X_t(x))^2\right) dx \quad \text{for } \beta < 1. \tag{1.3}$$

If $\beta = 1$, the above limit equals the zero measure almost surely. To construct the exactly scaling chaos measure at criticality $\beta = 1$, one has to perform a non-trivial normalization as follows:

$$d\mu_1(x) := \lim_{t\to\infty} \sqrt{t} \exp\left(X_t(x) - \frac{1}{2}\mathbb{E}\,(X_t(x))^2\right) dx, \tag{1.4}$$

where the limit now exists in probability.

The need of a nontrivial normalisation at the critical parameter value in (1.4) has been observed in many analogous situations before, e.g. [8, 33]. A convergence result analogous to (1.4) was proven by Aidekon and Shi in the important work [2] in the case of Mandelbrot chaos measures that can be thought of as a discrete analogue of continuous chaos. Independently C. Webb [31] obtained the corresponding result (with convergence in distribution) for the Gaussian cascades ([2] and [31] considered the total mass, but the convergence of the measures can then be verified without too much work). Finally, Duplantier, Rhodes, Sheffield and Vargas [10, 12] established (1.4) for a class of continuous Gaussian chaos measures including the exactly scaling one. We refer to [25, 11] for a much more thorough discussion of chaos measures and their applications, as well as for further references on the topic.

An important issue is to understand when the obtained chaos measure is independent of the choice of the approximating fields X_n. As mentioned before, Kahane's seminal work contained some results in this direction. Robert and Vargas [26] addressed the uniqueness question in the case of subcritical log-correlated fields (1.1) for convolution approximations $X_n = \phi_{\varepsilon_n} * X$. Duplantier's and Sheffield's paper [14] gives uniqueness results for particular approximations of the 2-dimensional GFF. More general results developing the method of [26] are contained in the review [25] due to Rhodes and Vargas, whose conditions are very similar to ours. In [9, 19][1] the method is also applied for a class of convolution approximations of the critical chaos. Another approach is contained in the paper of Shamov [28]. The techniques of the latter paper are based on an interesting new characterisation of chaos measures, which produces strong results but is applicable only in the subcritical range. Finally, in the paper [5] Berestycki provides an elegant and simple treatment of convolution approximations, again in the subcritical regime.

In the present paper we develop a new approach to the uniqueness question, which gives a simple proof of uniqueness in the subcritical regime, but more importantly it also applies to the case of critical chaos. Our idea uses a specifically tailored auxiliary field added to the original field in order to obtain comparability directly from Kahane's convexity inequality, and the choice is made so that in the limit the effect of the auxiliary field vanishes. The approach is outlined before the actual proof in the beginning of Section 3. One obtains a unified result that applies in general to chaos measures obtained via an arbitrary normalization, the only requirement is that the chaos measure is non-atomic almost surely. Therefore, our results apply also to a class of chaos measures that lie between the critical and supercritical ones, which one expects to be useful in the study of finer properties of the critical chaos itself.

Our basic result considers the following situation: Let (X_n) and $(\widetilde{X}_n)$ be two sequences of Hölder-regular Gaussian fields (see Section 2 for the precise definition) on a compact doubling metric space $(\mathcal{T}, d)$. Assume that for each $n \geq 1$ we have a non-negative Radon reference measure ρ_n defined on $\mathcal{T}$. Define the measures

$$d\mu_n(x) := e^{X_n(x) - \frac{1}{2}\mathbb{E}\,[X_n(x)^2]}\, d\rho_n(x)$$

for all $n \geq 1$. The measures $\widetilde{\mu}_n$ are defined analogously by using the fields $\widetilde{X}_n$ instead.

Theorem 1.1. *Let $C_n(x,y)$ and $\widetilde{C}_n(x,y)$ be the covariance functions of the fields X_n and $\widetilde{X}_n$ respectively. Assume that the random measures $\widetilde{\mu}_n$ converge in distribution to an almost surely non-atomic random measure $\widetilde{\mu}$ on $\mathcal{T}$. Moreover, assume that the covariances C_n and $\widetilde{C}_n$ satisfy the following two conditions: There exists a constant $K > 0$ such that*

$$\sup_{x,y\in\mathcal{T}} |C_n(x,y) - \widetilde{C}_n(x,y)| \leq K < \infty \quad \textit{for all } n \geq 1, \tag{1.5}$$

and

$$\lim_{n\to\infty} \sup_{d(x,y)>\delta} |C_n(x,y) - \widetilde{C}_n(x,y)| = 0 \quad \textit{for every } \delta > 0. \tag{1.6}$$

[1] We would like to thank the anonymous referee for pointing out the latter article.

Then the measures μ_n converge in distribution to the same random measure $\widetilde{\mu}$.

Remark 1.2. For simplicity we have stated the above theorem and will give the proof in the setting of a compact space $\mathcal{T}$. Similar results are obtained for non-compact $\mathcal{T}$ by standard localization. For example assume that $\mathcal{T}$ has an exhaustion $\mathcal{T} = \bigcup_{n=1}^{\infty} K_n$ with compacts $K_1 \subset K_2 \subset \cdots \subset \mathcal{T}$, such that every compact $K \subset \mathcal{T}$ is eventually contained in some K_n. Then if the assumptions of Theorem 1.1 are valid for the restrictions to each K_n, the claim also holds for $\mathcal{T}$, where now weak convergence is defined using compactly supported test functions.

The proof of the above theorem is contained in Section 3, where it is also noted that one may somewhat loosen the condition (1.5), see Remark 3.6. We refer to Section 2 for precise definitions of convergence in the space of measures and other needed prerequisities.

Section 4 addresses the interesting question when the convergence in Theorem 1.1 can be lifted to convergence in probability (or in L^p). Theorem 4.4 below provides practical conditions for checking this when the convergence is known for some other approximation sequence that has a martingale structure – a condition which is often met in applications.

In Section 5 we discuss consequences for convolution approximations (see Corollaries 5.2 and 5.4). In addition to general results we consider both circular averages and convolution approximations of the Gaussian free field in dimension 2 (Corollary 5.8).

Finally, Section 6 illustrates the use of the results of the previous sections. This is done via taking a closer look at the fundamental critical chaos on the unit circle, obtained from the GFF defined via the Fourier series

$$X(x) = 2\sqrt{\log 2}A_0 + \sqrt{2}\sum_{k=1}^{\infty} k^{-1/2}\big(A_k \sin(2\pi kx) + B_k \cos(2\pi kx)\big) \quad \text{for} \quad x \in [0,1),$$

where the A_n, B_n are independent standard Gaussians. In [3] the corresponding subcritical Gaussian chaos was constructed using martingale approximates defined via the periodic hyperbolic white noise. We shall consider four different approximations of X:

1. $X_{1,n}$ is the approximation of X obtained by cutting the periodic hyperbolic white noise construction of X on the level $1/n$.
2. $X_{2,n}(x) = 2\sqrt{\log 2}A_0 + \sqrt{2}\sum_{k=1}^{n} k^{-1/2}\big(A_k \sin(2\pi kx) + B_k \cos(2\pi kx)\big)$ for $x \in [0,1)$.
3. $X_{3,n} = \phi_{1/n} * X$, where ϕ is a mollifier function defined on $\mathcal{T}$ that satisfies some weak conditions.
4. $X_{4,n}$ is obtained as the nth partial sum of a vaguelet decomposition of X.

Theorem 1.3. *For all $j = 1, \ldots, 3$ the random measures*

$$\sqrt{\log n}\exp\Big(X_{j,n}(x) - \frac{1}{2}\mathbb{E}\,(X_{j,n}(x))^2\Big)\,dx$$

converge as $n \to \infty$ in probability to the same nontrivial random measure μ_{1,S^1} on $\mathcal{T}$, which is the fundamental critical measure on $\mathcal{T}$. The convergence actually takes place in $L^p(\Omega)$ for every $0 < p < 1$. The same holds for the vaguelet decomposition $X_{4,n}$ with the normalization $\sqrt{n\log 2}$ instead of $\sqrt{\log n}$.

We refer to Section 6 for the precise definitions of the approximations used above. Theorem 1.3 naturally holds true in the subcritical case if above $X_{j,n}$ is replaced by $\beta X_{j,n}$ with $\beta \in (0,1)$, and one removes the factor $\sqrt{\log n}$. We denote the limit measure by μ_{β,S^1}.

2 Notation and basic definitions

A metric space is *doubling* if there exists a constant $M > 0$ such that any ball of radius $\varepsilon > 0$ can be covered with at most M balls of radius $\varepsilon/2$. In this work we shall always consider a doubling compact metric space $(\mathcal{T}, d)$. We denote by $\mathcal{M}^+$ the space of (positive) Radon measures on $\mathcal{T}$. The space $\mathcal{M}$ of real-valued Radon measures on $\mathcal{T}$ can be given the weak*-topology by interpreting it as the dual of $C(\mathcal{T})$. We then give $\mathcal{M}^+ \subset \mathcal{M}$ the subspace topology.

The space $\mathcal{M}^+$ is metrizable (which is not usually the case for the full space $\mathcal{M}$), for example by using the Kantorovich–Rubinstein metric defined by

$$d(m, m') := \sup \left\{ \int_{\mathcal{T}} f(x)\, d(m - m')(x) : f \colon \mathcal{T} \to \mathbb{R} \text{ is 1-Lipschitz} \right\}.$$

For a proof see [7, Theorem 8.3.2].

Let $\mathcal{P}(\mathcal{M}^+)$ denote the space of Radon probability measures on $\mathcal{M}^+$. One should note that Borel probability measures and Radon probability measures coincide in this situation, as well as in the case of $\mathcal{P}(\mathcal{T})$, since we are dealing with Polish spaces. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space. We call a measurable map $\mu \colon \Omega \to \mathcal{M}^+$ a *random measure* on $\mathcal{T}$. For a given random measure μ the push-forward measure $\mu_* \mathbb{P} \in \mathcal{P}(\mathcal{M}^+)$ is called the distribution of μ and we say that a family of random measures μ_n converges in distribution if the measures $\mu_{n*}\mathbb{P}$ converge weakly in $\mathcal{P}(\mathcal{M}^+)$ (i.e. when evaluated against bounded continuous functions $\mathcal{P}(\mathcal{M}^+) \to \mathbb{R}$). In order to check the convergence in distribution, it is enough to verify that

$$\mu_n(f) := \int f(x)\, d\mu_n(x)$$

converges in distribution for every $f \in C(\mathcal{T})$, see e.g. [22, Theorem 16.16].

A stronger form of convergence is the following: We say that a sequence of random measures (μ_n) converges *weakly in L^p* to a random measure μ if for all $f \in C(\mathcal{T})$ the random variable $\int f(x)\, d\mu_n(x)$ converges in $L^p(\Omega)$ to $\int f(x)\, d\mu(x)$. This obviously implies the convergence $\mu_n \to \mu$ in distribution.

A (pointwise defined) Gaussian field X on $\mathcal{T}$ is a random process indexed by $\mathcal{T}$ such that $(X(t_1), \dots, X(t_n))$ is a multivariate Gaussian random variable for every $t_1, \dots, t_n \in \mathcal{T}$, $n \geq 1$. We will assume that all of our Gaussian fields are centered unless otherwise stated.

Definition 2.1. *A (centered) Gaussian field X on a compact metric space $\mathcal{T}$ is Hölder-regular if the map $(x, y) \mapsto \sqrt{\mathbb{E}\,|X(x) - X(y)|^2}$ is α-Hölder continuous on $\mathcal{T} \times \mathcal{T}$ for some $\alpha > 0$.*

Lemma 2.2. *The realizations of any Hölder-regular Gaussian field on $\mathcal{T}$ can be chosen to be almost surely β-Hölder continuous with some $\beta > 0$.*

Proof. This is an immediate consequence of Dudley's theorem (See for instance [1, Theorem 1.3.5].) and the fact that our space is doubling. □

Remark 2.3. By Dudley's theorem the conclusion of Lemma 2.2 would be valid under much less restrictive assumptions on the covariance, and most of the results of the present paper could be reformulated accordingly.

Assume that we are given a sequence of Hölder-regular Gaussian fields (X_n) on $\mathcal{T}$ and also a sequence of measures $\rho_n \in \mathcal{M}^+$. Define for all $n \geq 1$ a random measure $\mu_n \colon \Omega \to \mathcal{M}^+$ by setting

$$\mu_n(f) := \int_{\mathcal{T}} f(x) e^{X_n(x) - \frac{1}{2}\mathbb{E}\,[X_n(x)^2]}\, d\rho_n(x), \tag{2.1}$$

for all $f \in C(\mathcal{T})$. In the case where the measures μ_n converge in distribution to a random measure $\mu\colon \Omega \to \mathcal{M}^+$, we call μ a *Gaussian multiplicative chaos* (GMC) associated with the families X_n and ρ_n. We call the sequence of measures ρ_n a *normalizing sequence*. In the standard models of subcritical and critical chaos the typical choices are $\rho_n := \lambda$ and $\rho_n := C\sqrt{n}\lambda$ (or $\rho_n := C\sqrt{\log n}\lambda$), respectively, where λ stands for the Lebesgue measure.

Unless otherwise stated, when comparing the limits of two sequences of random measures (μ_n) and $(\widetilde{\mu}_n)$, we will always use the same normalizing sequence (ρ_n) to construct both μ_n and $\widetilde{\mu}_n$.

Lastly we recall the following fundamental convexity inequality due to Kahane [20].

Lemma 2.4. *Assume that X and Y are two Hölder-regular fields such that the covariances satisfy $C_X(s,t) \geq C_Y(s,t)$ for all $s,t \in \mathcal{T}$. Then for every concave function $f\colon [0,\infty) \to [0,\infty)$ we have*

$$\mathbb{E}\left[f\Big(\int_{\mathcal{T}} e^{X(t)-\frac{1}{2}\mathbb{E}\,[X(t)^2]}\,d\rho(t)\Big)\right] \leq \mathbb{E}\left[f\Big(\int_{\mathcal{T}} e^{Y(t)-\frac{1}{2}\mathbb{E}\,[Y(t)^2]}\,d\rho(t)\Big)\right]$$

for all $\rho \in \mathcal{M}^+$.

3 Convergence and uniqueness: Proof of Theorem 1.1

In this section we prove Theorem 1.1. The simple idea of the proof is as follows: We construct a sequence of auxiliary fields Y_ε (see especially Lemma 3.5) that we add on top of the fields X_n in order to ensure that the covariance of $X_n + Y_\varepsilon$ dominates the covariance of $\widetilde{X}_n$ pointwise. The fields Y_ε become fully decorrelated as $\varepsilon \to 0$, and their construction relies on the non-atomicity of the random measure $\widetilde{\mu}$. After these preparations one may finish by a rather standard application of Kahane's convexity inequality (Lemma 2.4).

The next two lemmata are almost folklore, but we provide proofs for completeness.

Lemma 3.1. *Let (μ_n) be a tight sequence of random measures. Then there exists a function $h\colon [0,\infty) \to [0,\infty)$ that has the following properties:*

1. *functions h, h^2 and h^4 are increasing and concave with $h(0) = 0$ and $\lim_{x\to\infty} h(x) = \infty$,*
2. *h satisfies $\min(1,x)h(y) \leq h(xy) \leq \max(1,x)h(y)$, and*
3. *$\sup_{n\geq 1} \mathbb{E}\, h(\mu_n(\mathcal{T}))^4 < \infty$.*

Proof. First of all, by the definition of tightness one may easily pick an increasing $g\colon [0,\infty) \to [1,\infty)$ with $\lim_{x\to\infty} g(x) = \infty$ such that $\sup_{n\geq 1} \mathbb{E}\,[g(\mu_n(\mathcal{T}))] < \infty$. Namely, let $0 = t_0 \leq t_1 \leq t_2 \leq \ldots$ be an increasing sequence of real numbers such that $\sup_{n\geq 1} \mathbb{P}[\mu_n(\mathcal{T}) \geq t_k] \leq k^{-2}$ for all $k \geq 1$ and set $g(x) = \sum_{k=0}^\infty \chi_{[t_k,\infty)}$. One may choose a concave function $\widetilde{h}$ that is majorized by g and satisfies both $\widetilde{h}(0) = 0$ and $\lim_{x\to\infty} h(x) = \infty$. Finally, set $h(x) := (\widetilde{h}(x))^{1/4}$. Condition (3) follows, and (2) is then automatically satisfied by concavity. Since compositions of non-negative concave functions remain concave we obtain (1) as well. ☐

Lemma 3.2. *For $n \geq 1$ let X_n and $\widetilde{X}_n$ be Hölder-regular Gaussian fields on $\mathcal{T}$ with covariance functions $C_n(x,y)$ and $\widetilde{C}_n(x,y)$. Define the random measures μ_n and $\widetilde{\mu}_n$ using the fields X_n and $\widetilde{X}_n$, respectively. Assume that there exists a constant $K > 0$ such that*

$$\sup_{x,y\in\mathcal{T}} (\widetilde{C}_n(x,y) - C_n(x,y)) \leq K < \infty$$

for all $n \geq 1$ and that the family $(\widetilde{\mu}_n)$ is tight (in $\mathcal{P}(\mathcal{M}^+)$). Then also the family (μ_n) is tight.

Proof. By the Banach–Alaoglu theorem it is enough to check that

$$\lim_{u\to\infty} \sup_{n\geq 1} \mathbb{P}[\mu_n(\mathcal{T}) > u] = 0.$$

Since $\lim_{u\to\infty} h(u) = \infty$, it suffices to verify that $\sup_{n\geq 1} \mathbb{E}\, h(\mu(\mathcal{T})) < \infty$, where h is the concave function given by Lemma 3.1 for the tight sequence $\widetilde{\mu}_n$. Pick an independent standard Gaussian G. By our assumption the covariance of the field $X_n' := X_n + K^{1/2}G$ dominates that of the field $\widetilde{X}_n$, and if the random measure μ_n' is defined by using the field X_n', we obtain by Kahane's concavity inequality

$$\mathbb{E}\,(h(\mu_n'(\mathcal{T})))^2 \leq \mathbb{E}\,(h(\widetilde{\mu}_n(\mathcal{T})))^2 \leq c \quad \text{for any } n \geq 1$$

for some constant $c > 0$ not depending on n.

Since $\mu_n' = e^{K^{1/2}G - K/2}\mu_n$ the properties (2) and (3) of Lemma 3.1 enable us to estimate for all $n \geq 1$ that

$$\begin{aligned}\mathbb{E}\, h(\mu_n(\mathcal{T})) &= \mathbb{E}\; h(e^{-K^{1/2}G+K/2}\mu_n'(\mathcal{T})) \leq \mathbb{E}\left(\max(1, e^{-K^{1/2}G+K/2})h(\mu_n'(\mathcal{T}))\right)\\ &\leq \left(\mathbb{E}\,(\max(1, e^{-K^{1/2}G+K/2}))^2\right)^{1/2}\left(\mathbb{E}\,(h(\widetilde{\mu}_n(\mathcal{T})))^2\right)^{1/2} \leq c'\sqrt{c},\end{aligned}$$

for some $c' > 0$. □

Our proof of Theorem 1.1 is based on the following two lemmas.

Lemma 3.3. *Let (X_n) and $(\widetilde{X}_n)$ be two sequences of Hölder-regular Gaussian fields on $\mathcal{T}$. Assume that there exists a constant $K > 0$ such that the covariances satisfy*

$$\sup_{x,y\in\mathcal{T}} |\widetilde{C}_n(x,y) - C_n(x,y)| \leq K < \infty$$

for all $n \geq 1$. Assume also that both of the corresponding sequences of random measures (μ_n) and $(\widetilde{\mu}_n)$ converge in distribution to measures μ and $\widetilde{\mu}$ respectively, and that $\widetilde{\mu}$ is almost surely non-atomic. Then also μ is almost surely non-atomic.

Proof. Let G be an independent centered Gaussian random variable with variance $\mathbb{E}\, G^2 = K$. Then the covariance of the field $X_n + G$ dominates that of the field $\widetilde{X}_n$. Define a field $U_n(x,y) := X_n(x) + X_n(y) + 2G$ on the product space $\mathcal{T}\times\mathcal{T}$. Its covariance is given by

$$\begin{aligned}\mathbb{E}\,[U_n(x,y)U_n(x',y')] = \mathbb{E}\,[X_n(x)X_n(x')] + \mathbb{E}\,[X_n(y)X_n(y')] + \mathbb{E}\,[X_n(x)X_n(y')]\\ + \mathbb{E}\,[X_n(y)X_n(x')] + 4K,\end{aligned}$$

and therefore dominates the covariance of the field $V_n(x,y) := \widetilde{X}_n(x) + \widetilde{X}_n(y)$ given by

$$\begin{aligned}\mathbb{E}\,[V_n(x,y)V_n(x',y')] = \mathbb{E}\,[\widetilde{X}_n(x)\widetilde{X}_n(x')] + \mathbb{E}\,[\widetilde{X}_n(y)\widetilde{X}_n(y')] + \mathbb{E}\,[\widetilde{X}_n(x)\widetilde{X}_n(y')]\\ + \mathbb{E}\,[\widetilde{X}_n(y)\widetilde{X}_n(x')].\end{aligned}$$

For $\varepsilon > 0$, let

$$f_\varepsilon(x,y) := \max\left(0, 1 - \frac{|x-y|}{\varepsilon}\right)$$

be a continuous approximation of the characteristic function of the diagonal $\Delta := \{(x,x) : x \in \mathcal{T}\} \subset \mathcal{T}\times\mathcal{T}$. Define a measure ρ_n' on $\mathcal{T}\times\mathcal{T}$ by setting

$$d\rho_n'(x,y) = f_\varepsilon(x,y)e^{\mathbb{E}\,[X_n(x)X_n(y)]}d(\rho_n\otimes\rho_n)(x,y)$$

and let h be as in Lemma 3.1. Then by Kahane's convexity inequality applied to the fields U_n and V_n w.r.t. the measure ρ_n' on the product space $\mathcal{T} \times \mathcal{T}$ we have

$$\begin{aligned}
&\mathbb{E}\, h((\mu_n \otimes \mu_n)(f_\varepsilon)) \\
&= \mathbb{E}\, h\Big(\int_{\mathcal{T}\times\mathcal{T}} f_\varepsilon(x,y) e^{U_n(x,y) - 2G - \frac{1}{2}\mathbb{E}\, U_n(x,y)^2 + \mathbb{E}\, X_n(x)X_n(y) + 2K}\, d(\rho_n \otimes \rho_n)(x,y)\Big) \\
&\le \mathbb{E}\, \max(1, e^{2K-2G}) \mathbb{E}\, h\Big(\int e^{U_n(x,y) - \frac{1}{2}\mathbb{E}\, U_n(x,y)^2}\, d\rho_n'(x,y)\Big) \\
&\le \mathbb{E}\, \max(1, e^{2K-2G}) \mathbb{E}\, h\Big(\int e^{V_n(x,y) - \frac{1}{2}\mathbb{E}\, V_n(x,y)^2}\, d\rho_n'(x,y)\Big) \\
&\le \mathbb{E}\, \max(1, e^{2K-2G}) e^K \mathbb{E}\, h((\widetilde{\mu}_n \otimes \widetilde{\mu}_n)(f_\varepsilon(x,y))).
\end{aligned}$$

Above we applied Lemma 3.1 (2) twice. By letting $n \to \infty$ we obtain

$$\mathbb{E}\, h((\mu \otimes \mu)(\Delta)) \le \mathbb{E}\, h((\mu \otimes \mu)(f_\varepsilon)) \le C\mathbb{E}\, h((\widetilde{\mu} \otimes \widetilde{\mu})(f_\varepsilon)),$$

where $C = e^K \mathbb{E}\, \max(1, e^{2K-2G})$ is a constant that only depends on K. Letting $\varepsilon \to 0$ lets us conclude that $(\mu \otimes \mu)(\Delta) = 0$ almost surely, which entails that μ is non-atomic almost surely. □

Remark 3.4. One should note that the above proof is not valid as such if one just assumes that the dominance of the covariance is valid in one direction only. In a sense we perform both a convexity and a concavity argument while deriving the required inequality. We do not know whether this is a limitation of our proof, or whether there exists an example where one-sided bound is not enough.

Lemma 3.5. *Assume that the conditions of Theorem 1.1 hold. Then there exists a collection Y_ε $(0 < \varepsilon < 1)$ of Hölder-regular Gaussian fields on $\mathcal{T}$ such that for a fixed $0 < \varepsilon < 1$ the covariance of the field $X_n + Y_\varepsilon$ is pointwise larger than the covariance of the field $\widetilde{X}_n$ for all large enough n. Moreover, there exists a constant $C = C(K)$ depending only on the constant K appearing in* (1.5) *such that*

$$\mathbb{E}\, \Big| \int_{\mathcal{T}} e^{Y_\varepsilon(x) - \frac{1}{2}\mathbb{E}\,[Y_\varepsilon(x)^2]}\, d\lambda(x) - \lambda(\mathcal{T}) \Big|^2 \le$$
$$3\varepsilon^2 \lambda(\mathcal{T})^2 + C(\lambda \otimes \lambda)(\{(x,y) \in \mathcal{T} : |x-y| < 2\varepsilon\})$$

for any $\lambda \in \mathcal{M}^+$ and $\varepsilon \in (0,1)$.

Proof. Fix a sequence of independent standard Gaussian random variables A_i, $i \ge 1$, such that they are also independent of the fields X_n. Let $\varepsilon > 0$ and choose a maximal set of points $a_1, \dots, a_n$ in $\mathcal{T}$ such that $|a_i - a_j| \ge \varepsilon/2$ for all $1 \le i < j \le n$. Let B_i be the ball $B(a_i, \varepsilon)$. Then the balls B_i cover $\mathcal{T}$ and we may form a Lipschitz partition of unity $p_1, \dots, p_n$ with respect to these balls. That is, $p_1, \dots, p_n$ are non-negative Lipschitz continuous functions such that $p_i(x) = 0$ when $x \notin B(a_i, \varepsilon)$ and for all $x \in \mathcal{T}$ we have $\sum_{i=1}^n p_i(x) \equiv 1$.

Define the field $Z_\varepsilon(x)$ by setting

$$Z_\varepsilon(x) = \sum_{i=1}^n A_i \sqrt{p_i(x)},$$

whence the covariance of Z_ε is given by

$$C_\varepsilon(x,y) := \mathbb{E}\,[Z_\varepsilon(x) Z_\varepsilon(y)] = \sum_{i=1}^n \sqrt{p_i(x) p_i(y)}.$$

By the Cauchy-Schwartz inequality we see that

$$C_\varepsilon(x,y) \leq \sqrt{\sum_{i=1}^n p_i(x)}\sqrt{\sum_{i=1}^n p_i(y)} = 1$$

for all $x, y \in \mathcal{T}$. Futhermore $C_\varepsilon(x,x) = 1$ for all $x \in \mathcal{T}$.

We may now define the field $Y_\varepsilon(x) = \varepsilon G + \sqrt{K} Z_\varepsilon(x)$ where G is a standard Gaussian random variable independent of the fields Z_ε and X_n. The conditions (1.5) and (1.6) together with compactness yield that for all large enough n the covariance of the field $X_n + Y_\varepsilon$ is greater than the covariance of the field $\widetilde{X}_n$ at every point $(x,y) \in \mathcal{T} \times \mathcal{T}$.

Now a direct computation gives

$$\mathbb{E}\left|\int_{\mathcal{T}} e^{Y_\varepsilon(x) - \frac{1}{2}\mathbb{E}[Y_\varepsilon(x)^2]}\, d\lambda(x) - \lambda(\mathcal{T})\right|^2 = \int_{\mathcal{T}}\int_{\mathcal{T}} \left(e^{KC_\varepsilon(x,y)+\varepsilon^2} - 1\right) d\lambda(x)\, d\lambda(y).$$

Clearly when $|x-y| \geq 2\varepsilon$, we have $|x-a_i| + |y-a_i| \geq 2\varepsilon$, so one of x or y lies outside of B_i for every $1 \leq i \leq n$, which implies that $C_\varepsilon(x,y) = 0$. Therefore we have

$$\begin{aligned}
&\int_{\mathcal{T}}\int_{\mathcal{T}} (e^{KC_\varepsilon(x,y)+\varepsilon^2} - 1)\, d\lambda(x)\, d\lambda(y) \\
&= (e^{\varepsilon^2} - 1)(\lambda \otimes \lambda)(\{|x-y| \geq 2\varepsilon\}) + \int_{\{|x-y|<2\varepsilon\}} \left(e^{\varepsilon^2 + KC_\varepsilon(x,y)} - 1\right) d(\lambda \otimes \lambda)(x,y) \\
&\leq (e^{\varepsilon^2} - 1)(\lambda \otimes \lambda)(\{|x-y| \geq 2\varepsilon\}) + (e^{\varepsilon^2+K} - 1)(\lambda \otimes \lambda)(\{|x-y| < 2\varepsilon\}),
\end{aligned}$$

from which the claim follows, since $e^{\varepsilon^2} - 1 \leq 3\varepsilon^2$ for $0 < \varepsilon < 1$. □

Proof of Theorem 1.1. We will first assume that both sequences (μ_n) and $(\widetilde{\mu}_n)$ converge in distribution and show how to get rid of this condition at the end.

Let Y_ε be the independent field constructed as in Lemma 3.5. We may assume, towards notational simplicity, that our probability space has the product form $\Omega = \Omega_1 \times \Omega_2$, and for $(\omega_1, \omega_2) \in \Omega$ one has $X_n((\omega_1,\omega_2)) = X_n(\omega_1)$ and $\widetilde{X}_n((\omega_1,\omega_2)) = \widetilde{X}_n(\omega_1)$ together with $Y_\varepsilon((\omega_1,\omega_2)) = Y_\varepsilon(\omega_2)$ for all $\varepsilon > 0$. Let $\varphi\colon [0,\infty) \to [0,\infty)$ be a bounded, continuous and concave function. Then by Kahane's convexity inequality we have

$$\begin{aligned}
\mathbb{E}\left[\varphi\left(\int_T f(x) e^{X_n(x)+Y_\varepsilon(x) - \frac{1}{2}\mathbb{E}[X_n(x)^2] - \frac{1}{2}\mathbb{E}[Y_\varepsilon(x)^2]}\, d\rho_n(x))\right)\right] \leq \\
\mathbb{E}\left[\varphi\left(\int_{\mathcal{T}} f(x) e^{\widetilde{X}_n - \frac{1}{2}\mathbb{E}[\widetilde{X}_n(x)^2]}\, d\rho_n(x)\right)\right]
\end{aligned}$$

for all non-negative $f \in C(\mathcal{T})$. Since for all fixed $\omega_2 \in \Omega_2$, $Y_\varepsilon(\omega_2)(x) - \frac{1}{2}\mathbb{E}[Y_\varepsilon(x)^2]$ is a continuous function on $\mathcal{T}$, we see that

$$\begin{aligned}
\mathbb{E}_{\Omega_1}\left[\varphi\left(\int_{\mathcal{T}} f(x) e^{X_n(x)+Y_\varepsilon(x) - \frac{1}{2}\mathbb{E}[X_n(x)^2] - \frac{1}{2}\mathbb{E}[Y_\varepsilon(x)^2]}\, d\rho_n(x)\right)\right] \to \\
\mathbb{E}_{\Omega_1}\left[\varphi\left(\int_{\mathcal{T}} f(x) e^{Y_\varepsilon(x) - \frac{1}{2}\mathbb{E}[Y_\varepsilon(x)^2]}\, d\mu(x)\right)\right]
\end{aligned}$$

as $n \to \infty$. In particular we have by Fatou's lemma that

$$\begin{aligned}
&\mathbb{E}\left[\varphi\left(\int_{\mathcal{T}} f(x)e^{Y_\varepsilon(x)-\frac{1}{2}\mathbb{E}[Y_\varepsilon(x)^2]}\,d\mu(x)\right)\right] \qquad (3.1)\\
=&\mathbb{E}_{\Omega_2}\lim_{n\to\infty}\mathbb{E}_{\Omega_1}\left[\varphi\left(\int_{\mathcal{T}} f(x)e^{X_n(x)+Y_\varepsilon(x)-\frac{1}{2}\mathbb{E}[X_n(x)^2]-\frac{1}{2}\mathbb{E}[Y_\varepsilon(x)^2]}\,d\rho_n(x)\right)\right]\\
\le&\liminf_{n\to\infty}\mathbb{E}\left[\varphi\left(\int_{\mathcal{T}} f(x)e^{\widetilde{X}_n-\frac{1}{2}\mathbb{E}[\widetilde{X}_n(x)^2]}\,d\rho_n(x)\right)\right]\\
=&\mathbb{E}\left[\varphi\left(\int_{\mathcal{T}} f(x)\,d\widetilde{\mu}(x)\right)\right].
\end{aligned}$$

Accoding to Lemma 3.5, for almost every $\omega_1 \in \Omega_1$ we know that

$$g_\varepsilon := \int_{\mathcal{T}} f(x)e^{Y_\varepsilon(x)-\frac{1}{2}\mathbb{E}[Y_\varepsilon(x)^2]}\,d\mu(x) \underset{\varepsilon\to 0}{\longrightarrow} g := \int_{\mathcal{T}} f(x)\,d\mu(x) \qquad (3.2)$$

in $L^2(\Omega_2)$. We next note that for a suitable fixed sequence $\varepsilon_k \to 0$ this convergence also happens for almost every $\omega_2 \in \Omega_2$. By Lemma 3.5 we have the estimate

$$\|g_\varepsilon - g\|^2_{L^2(\Omega_2)} \le 3\varepsilon^2\|f\|^2_{C(\mathcal{T})}\mu(\mathcal{T})^2 + C\|f\|^2_{C(\mathcal{T})}(\mu\otimes\mu)(\{|x-y|<2\varepsilon\}) =: \xi_\varepsilon,$$

Choose the sequence ε_k so that

$$\mathbb{P}[\xi_{\varepsilon_k} > 4^{-k}] \le \frac{1}{k^2},$$

which is possible because $(\mu\otimes\mu)(\{(x,x) : x \in \mathcal{T}\}) = 0$ almost surely. By the Borel–Cantelli lemma there exists a random index $k_0(\omega_1) \ge 1$ such that with probability 1 we have

$$\|g_{\varepsilon_k} - g\|^2_{L^2(\Omega_2)} \le 4^{-k}$$

for all $k \ge k_0(\omega_1)$. Now a standard argument verifies the almost sure convergence in (3.2).

The almost sure convergence finally lets us to conclude for all non-negative $f \in C(T)$ and non-negative, bounded, continuous and concave φ that

$$\mathbb{E}\left[\varphi\left(\int_{\mathcal{T}} f(x)\,d\mu(x)\right)\right] \le \mathbb{E}\left[\varphi\left(\int_{\mathcal{T}} f(x)\,d\widetilde{\mu}(x)\right)\right].$$

Similar inequality also holds with the measures μ and $\widetilde{\mu}$ switched, so we actually have

$$\mathbb{E}\left[\varphi\left(\int_{\mathcal{T}} f(x)\,d\mu(x)\right)\right] = \mathbb{E}\left[\varphi\left(\int_{\mathcal{T}} f(x)\,d\widetilde{\mu}(x)\right)\right].$$

It is well known that this implies $\mu \sim \widetilde{\mu}$.

Let us now finally observe that one can drop the assumption that both families of measures converge. By Lemma 3.2 and Prokhorov's theorem we know that every subsequence μ_{n_k} has a further subsequence that converges in distribution to a random measure. Lemma 3.3 ensures that the limit measure of any converging sequence has almost surely no atoms, and hence by the previous part of the proof this limit must equal $\widetilde{\mu}$. This implies that the original sequence must converge to $\widetilde{\mu}$ as well. □

Remark 3.6. Our proof of Theorem 1.1 may be modified in a way that allows the conditions (1.5) and (1.6) to be somewhat relaxed. E.g. in the case of subcritical logarithmically correlated fields it is basically enough to have for $\varepsilon > 0$ the inequality

$$|C_n(s,t) - \widetilde{C}_n(s,t)| \le \varepsilon(1 + \log^+ \frac{1}{|s-t|})$$

for $n \ge n(\varepsilon)$. Analogous results exist also for the critical chaos, but in this case the specific conditions are heavily influenced by the approximation sequence X_n one uses.

4 Convergence in probability

In the previous section convergence was established in distribution, which often suffices, and the main focus was on the uniqueness of the limit. In the present section we establish the convergence also in probability, assuming that this is true for the comparison sequence $\widetilde{\mu}_n$, which is constructed using approximating sequence $(\widetilde{X}_n)$ that has independent increments. Convergence in probability in the subcritical case was also discussed in [28], and our Theorem 4.4 below can be seen as an alternative way to approach the question.

Here is an outline of our method: We assume that the sequence μ_n is defined using linear approximations R_nX of the field X (see Definition 4.3), and invoke Lemma 4.1 to prove the convergence in probability by showing that if g is any (random) function that depends only on $X_1, \ldots, X_k$ for some fixed $k \geq 1$, then we have the convergence in distribution $g\, d\mu_n \to g\, d\widetilde{\mu}$. To establish the latter convergence, we split the measure μ_n as

$$d\mu_n = e^{E_{k,n}} e^{X_k - \frac{1}{2}\mathbb{E}\,[X_k^2]} e^{R_n(X-X_k) - \frac{1}{2}\mathbb{E}\,[(R_n(X_n-X_k))^2]}\, d\rho_n,$$

where $E_{k,n}$ is a $\sigma(X_1, \ldots, X_k)$-measurable error resulting from the approximation that goes to 0 as $n \to \infty$. By applying Lemma 4.2 we then conclude that $g\, d\mu_n$ converges to $ge^{X_k - \frac{1}{2}\mathbb{E}\,[X_k^2]}\, d\nu_k$ in distribution, where ν_k is a random measure independent of $X_1, \ldots, X_k$. Finally, by using the convergence in probability of $\widetilde{\mu}_n$ we can write $\widetilde{\mu} = e^{X_k - \frac{1}{2}\mathbb{E}\,[X_k^2]}\, d\eta_k$ for a random measure η_k, also independent of $X_1, \ldots, X_k$, and Lemma 4.2 tells us that ν_k and η_k have the same distribution. This lets us conclude that $ge^{X_k - \frac{1}{2}\mathbb{E}\,[X_k^2]}\, d\nu_k \sim g\, d\widetilde{\mu}$.

Enough speculation, it is time to work.

Lemma 4.1. *Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ be an increasing sequence of sigma-algebras and denote $\mathcal{F}_\infty := \sigma(\bigcup_{j=1}^\infty \mathcal{F}_k) \subset \mathcal{F}$. Assume that the real random variables $X, X_1, X_2, \ldots$ satisfy: X and X_k are $\mathcal{F}_\infty$-measurable, and for any $\mathcal{F}_j$ measurable set E (with arbitrary $j \geq 1$) it holds that*

$$\chi_E X_k \xrightarrow{d} \chi_E X \qquad \textit{as} \quad k \to \infty. \tag{4.1}$$

Then $X_k \xrightarrow{P} X$ as $k \to \infty$.

Proof. We first verify that (4.1) remains true also if the set E is just $\mathcal{F}_\infty$-measurable. For that end define $h_j := \mathbb{E}\,(\chi_E | \mathcal{F}_j)$ and construct an $\mathcal{F}_\infty$-measurable approximation $E_j := h_j^{-1}((1/2, 1])$. The martingale convergence theorem yields that $\mathbb{P}(E_j \Delta E) \to 0$ as $j \to \infty$. Since the claim holds for each E_j, it also follows for the set E by a standard approximation argument.

Let us then establish the stated convergence in probability. Fix $\varepsilon > 0$ and pick $M > 0$ large enough so that $\mathbb{P}(|X| > M/2) \leq \varepsilon/2$, and such that $\mathbb{P}(|X| = M) = 0$. Then for some k_0 we have that $\mathbb{P}(|X_k| \geq M) \leq \varepsilon$ if $k \geq k_0$. Divide the interval $(-M, M]$ into non overlapping half open intervals $I_1, \ldots, I_\ell$ of length less than $\varepsilon/2$ and denote $E_j := X^{-1}(I_j)$ for $j = 1, \ldots, \ell$. In the construction we may assume that 0 is the center point of one of these intervals and $\mathbb{P}(X = a) = 0$ if a is an endpoint of any of the intervals. We fix j and apply condition (4.1) to deduce that $\chi_{E_j} X_k \xrightarrow{d} \chi_{E_j} X$ as $k \to \infty$. Assume first that $0 \notin I_j$. Then the Portmonteau theorem yields that $\lim_{k\to\infty} \mathbb{P}(\chi_{E_j} X_k \in I_j) = \mathbb{P}(\chi_{E_j} X \in I_j)$, or in other words

$$\mathbb{P}(\{X \in I_j\} \cap \{X_k \in I_j\}) \to \mathbb{P}(X \in I_j) \quad \text{as} \quad k \to \infty.$$

In particular, for large enough k we have that

$$\mathbb{P}\big(E_j \cap (|X - X_k| > \varepsilon)\big) \leq \frac{\varepsilon}{2\ell} \tag{4.2}$$

If $0 \in I_j$ we obtain in a similar vein that $\lim_{k\to\infty} \mathbb{P}(\chi_{E_j} X_k \in (I_j)^c) = \mathbb{P}(\chi_{E_j} X \in (I_j)^c) = 0$, or in other words $\mathbb{P}(\{X \in I_j\} \cap \{X_k \in I_j^c\}) \to 0$, so that we again get that $\mathbb{P}\big(E_j \cap (|X -$

$X_k| > \varepsilon\big) \le \frac{\varepsilon}{2\ell}$ for large enough k. By summing the obtained inequalities for $j = 1, \dots, \ell$ and observing that $\mathbb{P}(\bigcup_{k=1}^{\ell} E_k) > 1 - \varepsilon/2$ we deduce for large enough k the inequality $\mathbb{P}(|X - X_k| > \varepsilon) < \varepsilon$, as desired. □

Lemma 4.2. *Let X be a Hölder-regular Gaussian field on $\mathcal{T}$ that is independent of the random measures μ and ν on $\mathcal{T}$.*

- **(i)** *If $e^X \mu \sim e^X \nu$, then also $\mu \sim \nu$.*
- **(ii)** *If (μ_n) is a sequence of random measures such that the sequence $(e^X \mu_n)$ converges in distribution, then also the sequence (μ_n) converges in distribution.*

Proof. We will first show that if X is of the simple form Nf with N a standard Gaussian random variable and $f \in C(\mathcal{T})$, then the claim holds. To this end let us fix $g \in C(\mathcal{T})$ and consider the function $\varphi \colon \mathbb{R} \to \mathbb{C}$ defined by

$$\varphi(x) = \mathbb{E}\,[\exp\big(i \int e^{Nf} e^{-xf} g\, d\mu\big)] = \mathbb{E}\,[\exp\big(i \int e^{Nf} e^{-xf} g\, d\nu\big)].$$

Because N is independent of μ and ν, we may write

$$\varphi(x) = \int_{-\infty}^{\infty} \mathbb{E}\,[\exp(i \int e^{(y-x)f} g\, d\mu)] \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy.$$

By denoting $u(t) = \mathbb{E}\,[\exp(i \int e^{-tf} g\, d\mu)]$, $v(t) = \mathbb{E}\,[\exp(i \int e^{-tf} g\, d\nu)]$ and $h(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, we see that $\varphi(x) = (u * h)(x) = (v * h)(x)$. Because the Fourier transform of h is also Gaussian we deduce by taking convolutions that the Fourier transforms $\widehat{u}$ and $\widehat{v}$ coincide as Schwartz distributions. Since u and v are continuous, this implies that $u(x) = v(x)$ for all x. In particular setting $x = 0$ gives us

$$\mathbb{E}\,[\exp(i \int g\, d\mu)] = \mathbb{E}\,[\exp(i \int g\, d\nu)],$$

for all $g \in C(\mathcal{T})$, whence the measures μ and ν have the same distribution.

To deduce the general case, note that we have the Karhunen–Loève decomposition

$$X = \sum_{k=1}^{\infty} N_k f_k$$

where N_k are standard Gaussian random variables and $f_k \in C(\mathcal{T})$ for all $k \in \mathbb{N}$. Moreover the above series converges almost surely uniformly. (See for example [1, Theorem 3.1.2.].) By the first part of the proof we know that $e^{\sum_{k=n}^{\infty} N_k f_k} \mu$ and $e^{\sum_{k=n}^{\infty} N_k f_k} \nu$ have the same distribution for all $n \in \mathbb{N}$. By the dominated convergence theorem we have

$$\begin{aligned} \mathbb{E}\,[\exp(i \int g\, d\mu)] &= \lim_{n\to\infty} \mathbb{E}\,[\exp(i \int e^{\sum_{k=n}^{\infty} N_k f_k} g\, d\mu)] \\ &= \lim_{n\to\infty} \mathbb{E}\,[\exp(i \int e^{\sum_{k=n}^{\infty} N_k f_k} g\, d\nu)] = \mathbb{E}\,[\exp(i \int g\, d\nu)] \end{aligned}$$

for all $g \in C(\mathcal{T})$, which shows the claim.

The second part of the lemma follows from the first part. Since $\sup_{t\in\mathcal{T}} X(t) < \infty$ almost surely, one checks that the sequence (μ_n) inherits the tightness of the sequence $(e^X \mu_n)$. It is therefore enough to show that any two converging subsequences have the same limit. Indeed, assume that $\mu_{k_j} \to \mu$ and $\mu_{n_j} \to \nu$ in distribution. Then by independence we have $e^X \mu_{k_j} \to e^X \mu$ and $e^X \mu_{n_j} \to e^X \nu$, but by assumption the limits are equally distributed and hence also μ and ν have the same distribution. □

A typical example of a linear regularization process described in the following definition is given by a standard convolution approximation sequence. We denote by $C^\alpha(\mathcal{T})$ the Banach space of α-Hölder continuous functions on $\mathcal{T}$.

Definition 4.3. *Let (X_k) be a sequence of approximating fields on $\mathcal{T}$. We say that a sequence (R_n) of linear operators $R_n\colon \bigcup_{\alpha\in(0,1)} C^\alpha(\mathcal{T}) \to C(\mathcal{T})$ is a* linear regularization process *for the sequence (X_k) if the following properties are satisfied:*

1. *We have $\lim_{n\to\infty} \|R_n f - f\|_\infty = 0$ for all $f \in \bigcup_{\alpha\in(0,1)} C^\alpha(\mathcal{T})$.*
2. *The limit $R_n X := \lim_{k\to\infty} R_n X_k$ exists in $C(\mathcal{T})$ almost surely.*

Theorem 4.4. *Assume that the increments $\{X_{m+1} - X_m : m \geq 1\}$ of the approximating fields X_m are independent and that there is the convergence in probability*

$$d\widetilde{\mu}_n := e^{X_n - \frac{1}{2}\mathbb{E}\,[X_n^2]}\, d\rho_n \xrightarrow[n\to\infty]{P} \widetilde{\mu}. \tag{4.3}$$

Let R_n be some linear regularization process for the sequence X_k such that

$$e^{R_n X - \frac{1}{2}\mathbb{E}\,[(R_n X)^2]}\, d\rho_n \xrightarrow[n\to\infty]{d} \widetilde{\mu}.$$

Then also $d\mu_n = e^{R_n X - \frac{1}{2}\mathbb{E}\,[(R_n X)^2]}\, d\rho_n$ converges to $\widetilde{\mu}$ in probability.

Remark 4.5. As in Remark 1.2 the above theorem extends to the case of a non-compact $\mathcal{T}$ when the assumptions are suitably reinterpreted. In a particular application it is also enough to assume the condition (1) in Definition 4.3 for one suitable fixed value of $\alpha > 0$, if the exponent of the Hölder regularity of the approximating fields is known.

Proof. Define the filtration $\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$. First of all, since $e^{X_n - \frac{1}{2}\mathbb{E}\,[X_n^2]}\, d\rho_n$ converges to $\widetilde{\mu}$ in probability as $n \to \infty$, we also have

$$e^{X_n - X_k - \frac{1}{2}\mathbb{E}\,[(X_n - X_k)^2]}\, d\rho_n \xrightarrow[n\to\infty]{P} e^{-X_k + \frac{1}{2}\mathbb{E}\,[X_k^2]}\widetilde{\mu} \quad \text{for every } k \geq 1.$$

To see this, one uses that $\mathbb{E}\,[(X_n - X_k)^2] = \mathbb{E}\,[X_n^2] - \mathbb{E}\,[X_k^2]$ and considers almost surely converging subsequences, if necessary. We denote $\eta_k := e^{-X_k + \frac{1}{2}\mathbb{E}\,[X_k^2]}\widetilde{\mu}$.

Notice that $\mathbb{E}\,[(R_n X)(R_n X_k)] = \mathbb{E}\,[(R_n X_k)^2]$ by the independent increments and the definition of $R_n X$. We may thus write

$$\begin{aligned} d\mu_n &= e^{R_n X - \frac{1}{2}\mathbb{E}\,[(R_n X)^2]}\, d\rho_n \\ &= \Big[e^{R_n X_k - X_k + \frac{1}{2}\mathbb{E}\,[X_k^2 - (R_n X_k)^2]}\Big] e^{X_k - \frac{1}{2}\mathbb{E}\,[X_k^2]} e^{R_n(X - X_k) - \frac{1}{2}\mathbb{E}\,[(R_n(X - X_k))^2]}\, d\rho_n. \end{aligned} \tag{4.4}$$

Above on the right hand side the term in brackets is negligible as $n \to \infty$. To see this, we note first that $e^{R_n X_k - X_k}$ tends almost surely to the constant function 1 uniformly according to Definition 4.3(1). Moreover, $\mathbb{E}\,[X_k^2 - (R_n X_k)^2]$ tends to 0 in $C(\mathcal{T})$, since the field X_k takes values in a fixed $C^\gamma(\mathcal{T})$ for some $\gamma > 0$, and by the Banach–Steinhaus theorem $\sup_{n\geq 1} \|R_n\|_{C^\gamma(T)\to C(\mathcal{T})} < \infty$. Namely,

$$\begin{aligned} \|\mathbb{E}\,[X_k^2 - (R_n X_k)^2]\|_{C(\mathcal{T})} &\leq \mathbb{E}\,\|(X_k - R_n X_k)(X_k + R_n X_k)\|_{C(\mathcal{T})} \\ &\leq \mathbb{E}\Big[\|X_k - R_n X_k\|_{C(\mathcal{T})} \|X_k + R_n X_k\|_{C(\mathcal{T})}\Big] \\ &\lesssim \mathbb{E}\,\|X_k\|^2_{C^\gamma(\mathcal{T})}, \end{aligned}$$

whence the dominated convergence theorem applies, since $\|X_k\|_{C^\gamma(\mathcal{T})}$ has a super exponential tail by Fernique's theorem. All in all, invoking the assumption on the convergence of μ_n we deduce that

$$e^{X_k - \frac{1}{2}\mathbb{E}\,[X_k^2]} e^{R_n(X - X_k) - \frac{1}{2}\mathbb{E}\,[(R_n(X - X_k))^2]}\, d\rho_n \longrightarrow \widetilde{\mu} \tag{4.5}$$

in distribution as $n \to \infty$.

By Lemma 4.2 we thus have the distributional convergence

$$e^{R_n(X-X_k)-\frac{1}{2}\mathbb{E}\,[(R_n(X-X_k))^2]}\,d\rho_n \longrightarrow \nu_k \quad \text{as} \;\; n \to \infty,$$

where the limit ν_k may be assumed to be independent of $\mathcal{F}_k$. In particular, recalling (4.5) we deduce that $e^{X_k-\frac{1}{2}\mathbb{E}\,[X_k^2]}\nu_k$ has the same distribution as $\widetilde{\mu} = e^{X_k-\frac{1}{2}\mathbb{E}\,[X_k^2]}\eta_k$. Lemma 4.2 now verifies that $\nu_k \sim \eta_k$. In order to invoke Lemma 4.1, fix any $\mathcal{F}_k$ measurable bounded random variable g. Then g and X_k are independent of $X - X_k$, and we therefore have the distributional convergence

$$\begin{aligned} &g e^{X_k-\frac{1}{2}\mathbb{E}\,[X_k^2]} e^{R_n(X-X_k)-\frac{1}{2}\mathbb{E}\,[(R_n(X-X_k))^2]}\,d\rho_n \\ &\underset{n\to\infty}{\longrightarrow} \; g e^{X_k-\frac{1}{2}\mathbb{E}\,[X_k^2]}\,d\nu_k \;\; \sim \;\; g e^{X_k-\frac{1}{2}\mathbb{E}\,[X_k^2]}\,d\eta_k \; = \; g\,d\widetilde{\mu}, \end{aligned} \tag{4.6}$$

where the second last equality followed by independence. Finally, again by the negligibility of the term $e^{R_nX_k-X_k}e^{-\frac{1}{2}\mathbb{E}\,[X_k^2-(R_nX_k)^2]}$ and using (4.4) we see that (4.6) in fact entails the convergence of $g\,d\mu_n$ to $g\,d\widetilde{\mu}$ in distribution. At this stage Lemma 4.1 applies and the desired claim follows. $\square$

Remark 4.6. In the previous theorem it was crucial that we already have an approximating sequence of fields along which the corresponding chaos converges in probability. In general if one only assumes convergence in distribution in (4.3), one may not automatically expect that it is possible to lift the convergence to that in probability, even for natural approximating fields. However, for most of the standard constructions of subcritical chaos this problem does not occur, as we have even almost sure convergence in (4.3) due to the martingale convergence theorem.

5 Convolution approximations

In this section we provide a couple of useful results for dealing with convolution approximations, $\star$-scale invariant fields and circular averages of 2-dimensional Gaussian fields. We also note that the results can be applied to a 2-dimensional Gaussian free field in a domain.

The next lemma and its corollaries show that any two convolution approximations (with some regularity) applied to log-normal chaos stay close to each other in the sense of Theorem 1.1.

Lemma 5.1. *Let $\varphi, \psi \colon \mathbb{R}^d \to \mathbb{R}$ satisfy $\int \varphi(x)\,dx = \int \psi(x)\,dx = 1$ and $|\varphi(x)|, |\psi(x)| \leq C(1+|x|)^{-(d+\delta)}$ for all $x \in \mathbb{R}^d$ with some constants $C, \delta > 0$. Then if $u \in BMO(\mathbb{R}^d)$, we have*

$$|(\varphi_\varepsilon * u)(x) - (\psi_\varepsilon * u)(x)| \leq K$$

for some constant $K > 0$ not depending on ε.

Proof. One can use the mean zero property and decay of $\varphi - \psi$ together with a standard BMO-type estimate [17, Proposition 7.1.5.] to see that for any $\varepsilon > 0$ we have

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} (\varphi_\varepsilon - \psi_\varepsilon)(t) u(x-t)\,dt \right| \\ &= \left| \int_{\mathbb{R}^d} (\varphi - \psi)(t) \Big(u(\varepsilon(x-t)) - \fint_{B(0,1)} u(\varepsilon(x-s))\,ds \Big)\,dt \right| \\ &\leq \int_{\mathbb{R}^d} \frac{|u(\varepsilon(x-t)) - \fint_{B(0,1)} u(\varepsilon(x-s))\,ds|}{(1+|t|)^{d+\delta}}\,dt \\ &\leq C_{d,\delta} \|u(\varepsilon(x-\cdot))\|_{BMO} = C_{d,\delta}\|u\|_{BMO}. \end{aligned}$$

$\square$

Corollary 5.2. *Let $f(x,y) = 2d\beta^2 \log^+ \frac{1}{|x-y|} + g(x,y)$ be a covariance kernel of a distribution valued field X defined on $\mathbb{R}^d$. Here g is a bounded uniformly continuous function. Assume that φ and ψ are two locally Hölder continuous convolution kernels in $\mathbb{R}^d$ that satisfy the conditions of Lemma 5.1. Let (ε_n) be a sequence of positive numbers ε_n converging to 0. Then the approximating fields $X_n := \varphi_{\varepsilon_n} * X$ and $\widetilde{X}_n := \psi_{\varepsilon_n} * X$ satisfy the conditions* (1.5) *and* (1.6) *of Theorem 1.1.*

Proof. The function $\ell(x) := 2d\beta^2 \log^+ \frac{1}{|x|}$ belongs to $BMO(\mathbb{R}^d)$ since $\log|x| \in BMO(\mathbb{R}^d)$, see for example [17, Example 7.1.3]. One computes that the covariance of $\varphi_\varepsilon * X$ equals

$$\int\int \varphi_\varepsilon(x-t)\varphi_\varepsilon(y-s)\ell(t-s)\,dt\,ds + \int\int \varphi_\varepsilon(x-t)\varphi_\varepsilon(y-s)g(t,s)\,dt\,ds.$$

Because g is bounded and uniformly continuous the second term goes to $g(x,y)$ uniformly, so we may without loss of generality assume that $g(x,y) = 0$. The first term equals $(\varphi_\varepsilon * \varphi_\varepsilon(-\cdot) * \ell)(x-y)$, so the condition (1.5) follows from Lemma 5.1 applied to the convolution kernels $\varphi * \varphi(-\cdot)$ and $\psi * \psi(-\cdot)$. Here one easily checks that also $\varphi * \varphi(-\cdot)$ satisfies the conditions of Lemma 5.1 and that $(\varphi * \varphi(-\cdot))_\varepsilon = \varphi_\varepsilon * \varphi_\varepsilon(-\cdot)$. Finally, the condition (1.6) is immediate. □

Remark 5.3. One may easily state localized versions of the above corollary.

Corollary 5.4. *Assume that $f(x,y) = 2\beta^2 \log^+ \frac{1}{2|\sin(\pi(x-y))|} + g(x,y)$ is the covariance of a* (*distribution valued*) *field X on the unit circle. Here g is a bounded continuous function that is 1-periodic in both variables x and y and we have identified the unit circle with $\mathbb{R}/\mathbb{Z}$. Assume that φ and ψ are two locally Hölder continuous convolution kernels in $\mathbb{R}$ that satisfy the conditions of Lemma 5.1, and let (ε_n) be a sequence of positive numbers ε_n converging to 0. Then the approximating fields $X_n := \varphi_{\varepsilon_n} * X$ and $\widetilde{X}_n := \psi_{\varepsilon_n} * X$ satisfy the conditions* (1.5) *and* (1.6) *of Theorem 1.1.*

Remark 5.5. Above when defining the approximating fields X_n we assume that X stands for the corresponding periodized field on $\mathbb{R}$ and the fields X_n will then automatically be periodic so that they also define fields on the unit circle.

Proof. One easily checks that $\ell(x) = 2\beta^2 \log^+ \frac{1}{2|\sin(\pi x)|}$ is in $BMO(\mathbb{R})$. The rest of the proof is analogous to the one of the previous corollary. □

The previous result showed that different convolution approximations lead to the same chaos. In turn, in order to show that a single convolution approximation converges to the desired chaos, one may often compare the convolution approximation directly to a martingale approximation field used originally to define the chaos. As an example of this, we show that the convolutions of $\star$-scale invariant fields are comparable (in the sense of Theorem 1.1) with the natural approximating fields arising from the $\star$-scale decomposition. This also extends the convergence of the critical chaos in [12] to convolution approximations.

Lemma 5.6. *Let $k\colon [0,\infty) \to \mathbb{R}$ be a compactly supported and positive definite C^1-function with $k(0) = 1$. Define the $\star$-scale invariant field X on $\mathbb{R}^d$, whose covariance is (formally) given by*

$$\mathbb{E}\, X(x)X(y) = \int_1^\infty \frac{k(u|x-y|)}{u}\,du. \tag{5.1}$$

Moreover, let φ be a convolution kernel satisfying the conditions of Corollary 5.2. Then the approximating fields $X_n := \varphi_{e^{-n}} * X$ and the fields $\widetilde{X}_n$ whose covariance is given by

$$\mathbb{E}\, \widetilde{X}_n(x)\widetilde{X}_n(y) = \int_1^{e^n} \frac{k(u|x-y|)}{u}\,du$$

satisfy the conditions (1.5) *and* (1.6) *of Theorem 1.1.*

Proof. One may easily check that the covariance in (5.1) is of the form

$$\log^+ \frac{1}{|x-y|} + g(x,y),$$

and therefore by Corollary 5.2 it is enough to show the claim for one mollifier φ. In particular, we may without generality assume that the support of φ is contained in $B(0,1/2)$ and that φ is a symmetric non-negative function. A short calculation shows that we have

$$\mathbb{E}\, X_n(0)X_n(x) = \int_1^\infty \frac{(\varphi_{e^{-n}} * \varphi_{e^{-n}} * k(u|\cdot|))(|x|)}{u}\, du.$$

Let $\psi = \varphi * \varphi$. Then the support of ψ is contained in $B(0,1)$ and $\psi_{e^{-n}} = \varphi_{e^{-n}} * \varphi_{e^{-n}}$. Thus we get

$$\begin{aligned} &\mathbb{E}\, X_n(0)X_n(x) - \mathbb{E}\, \widetilde{X}_n(0)\widetilde{X}_n(x) \\ &= \int_1^{e^n} \frac{(\psi_{e^{-n}} * k(u|\cdot|))(x) - k(u|x|)}{u}\, du + \int_{e^n}^\infty \frac{(\psi_{e^{-n}} * k(u|\cdot|))(x)}{u}\, du. \end{aligned}$$

Fix $R > 0$ so that the support of k is contained in $[0,R]$. Then we have

$$\begin{aligned} |(\psi_{e^{-n}} * k(u|\cdot|))(x) - k(u|x|)| &\le \int_{B(x,e^{-n})} \psi_{e^{-n}}(x-s)|k(u|s|) - k(u|x|)|\, ds \\ &\le \begin{cases} 0, & \text{if } (|x| - e^{-n})u > R \\ u\|k'\|_\infty e^{-n}, & \text{otherwise} \end{cases} \end{aligned}$$

We also have the bound

$$\begin{aligned} |(\psi_{e^{-n}} * k(u|\cdot|))(x)| &\le \|k\|_\infty \int_{B(0,R/u)} \psi_{e^{-n}}(x-s)\, ds \\ &\le \begin{cases} 0, & \text{if } (|x| - e^{-n})u > R \\ C\|k\|_\infty \|\psi\|_\infty \frac{R^d}{u^d} e^{nd}, & \text{otherwise} \end{cases} \end{aligned}$$

for some constant $C > 0$. Using just the upper bounds of these estimates for all x we get

$$\Big| \int_1^{e^n} \frac{|(\psi_{e^{-n}} * k(u|\cdot|))(x) - k(u|x|)}{u}\, du \Big| \le \|k'\|_\infty$$

and

$$\Big| \int_{e^n}^\infty \frac{(\psi_{e^{-n}} * k(u|\cdot|))(x)}{u}\, du \Big| \le C\|k\|_\infty \|\psi\|_\infty R^d/d,$$

verifying (1.5). Assume then that $\delta > 0$ is fixed and $|x| > \delta$. Then for large enough n we have that $e^{-n} < \delta/2$ and

$$\Big| \int_1^{e^n} \frac{|(\psi_{e^{-n}} * k(u|\cdot|))(x) - k(u|x|)}{u}\, du \Big| \le \int_1^{\frac{2R}{\delta}} \|k'\|_\infty e^{-n}\, du \to 0$$

and

$$\int_{e^n}^\infty \frac{(\psi_{e^{-n}} * k(u|\cdot|))(x)}{u}\, du = 0,$$

showing (1.6). □

Finally, we state a result for circle averages of 2-dimensional Gaussian fields.

Lemma 5.7. *Let X be a two dimensional Gaussian field with covariance of the form $\mathbb{E}\, X(x)X(y) = 4\beta^2 \log^+ \frac{1}{|x-y|} + g(x,y)$, where g is continuous and bounded. Let $X_\varepsilon(x) = \frac{1}{2\pi}\int_0^{2\pi} X(x+\varepsilon e^{i\theta})\,d\theta$ and let φ be a convolution kernel satisfying the conditions of Corollary 5.2. Then the approximating fields $X_n := \varphi_{e^{-n}} * X$ and the fields $\widetilde{X}_n := \frac{1}{2\pi}\int_0^{2\pi} X(x+e^{-n+i\theta})\,d\theta$ satisfy the conditions* (1.5) *and* (1.6) *of Theorem 1.1.*

Proof. We may compute

$$\mathbb{E}\,\widetilde{X}_n(x)\widetilde{X}_n(y) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \Big(4\beta^2 \log^+ \frac{1}{|x+e^{-n+is}-y-e^{-n+it}|} + g(x+e^{-n+is}, y+e^{-n+it})\Big)\,ds\,dt.$$

Clearly we can assume that $g=0$, since that part of the integral is bounded by a constant and converges uniformly. Moreover, we may assume that $|x-y| \le \frac{1}{2}$, since the integral converges uniformly to the right value as $n\to\infty$ when $|x-y|\ge \frac{1}{2}$. Thus we may write

$$\mathbb{E}\,\widetilde{X}_n(x)\widetilde{X}_n(y) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} 4\beta^2 \log \frac{1}{|x+e^{-n+is}-y-e^{-n+it}|}$$

for n large enough so that $|x+e^{-n+is}-y-e^{-n+it}|<1$. Now if $|x-y|>2e^{-n}$, then by invoking the harmonicity of the logarithm and using the mean value principle twice, we have

$$\mathbb{E}\,\widetilde{X}_n(x)\widetilde{X}_n(y) = 4\beta^2\log\frac{1}{|x-y|}.$$

On the other hand if $|x-y|\le 2e^{-n}$, then we may write

$$\mathbb{E}\,\widetilde{X}_n(x)\widetilde{X}_n(y) = n + \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} 4\beta^2 \log\frac{1}{|e^n(x-y)+e^{is}-e^{it}|},$$

where the integrand on the right hand side is bounded from below, and boundedness from above of the whole integral follows since the inner integral contains at most a logarithmic singularity, which is integrable. Thus we have shown that

$$\mathbb{E}\,\widetilde{X}_n(x)\widetilde{X}_n(y) = \begin{cases} n+O(1), & \text{if } |x-y|\le 2e^{-n} \\ 4\beta^2\log^+\frac{1}{|x-y|} + g(x,y) + o(1), & \text{if } |x-y| > 2e^{-n}. \end{cases}$$

This is enough to show the claim, since it is easy to check that certain convolution kernels φ yield approximations with similar covariance structure. □

We then very briefly note that the above results can be applied to the 2-dimensional Gaussian free field and its variants. We refer to the paper [12] for the definition of the massless free field (MFF) and a Gaussian free field (GFF) in a bounded domain.

Corollary 5.8. *Let X be the MFF, or a GFF in some planar domain with Dirichlet boundary conditions. Then the critical chaos defined via convolution approximations (naturally one needs to localize in the case of GFF) of X exists and is independent of the convolution kernel used. The same applies to the circle averages.*

Proof. The MFF is of the $\star$-scale invariant form, so our result applies directly. In the case of a GFF, we may write X as a smooth perturbation of the MFF (see [12]), whence the claim follows easily. □

Remark 5.9. We note that Theorem 4.4 often applies for convolution approximations. Especially it can be easily localized and it works for the MFF and GFF, including circular average approximations. The verification of the latter fact is not difficult and we omit it here.

Remark 5.10. Convergence of convolution approximations for the critical GFF in the unit circle have also been proven in [19]. The method used there is 'interpolation' of Gaussian fields X_1 and X_2 by the formula $\sqrt{t}X_1 + \sqrt{1-t}X_2$, used already in [26]. It is not immediately clear how far beyond convolution approximations this approach can be extended.

6 An application (Proof of Theorem 1.3)

The main purpose of this chapter is to prove Theorem 1.3 and explain carefully the approximations mentioned there. For the reader's convenience we try to be fairly detailed, although some parts of the material are certainly well-known to the experts.

We start by defining the approximation $X_{2,n}$ of the restriction of the free field on the unit circle $S^1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$. Following [3] recall that the trace of the Gaussian free field on the unit circle (identified with $\mathbb{R}/\mathbb{Z}$) is defined to be the Gaussian field[2]

$$X(x) = 2\sqrt{\log 2}G + \sqrt{2}\sum_{k=1}^{\infty}\Big(\frac{A_k}{\sqrt{k}}\cos(2\pi kx) + \frac{B_k}{\sqrt{k}}\sin(2\pi kx)\Big), \tag{6.1}$$

where A_k, B_k and G are independent standard Gaussian random variables. The field X is distribution valued and its covariance (more exactly, the kernel of the covariance operator) can be calculated to be

$$\mathbb{E}\left[X(x)X(y)\right] = 4\log(2) + 2\log\frac{1}{2|\sin(\pi(x-y))|}. \tag{6.2}$$

A natural approximation of X is then obtained by considering the partial sum of the Fourier series

$$X_{2,n}(x) := 2\sqrt{\log 2}G + \sqrt{2}\sum_{k=1}^{n}\Big(\frac{A_k}{\sqrt{k}}\cos(2\pi kx) + \frac{B_k}{\sqrt{k}}\sin(2\pi kx)\Big).$$

Another way to get hold of this covariance is via the periodic upper half-plane white noise expansion that we define next – recall that the non-periodic hyperbolic white noise W and the hyperbolic area measure m_{hyp} were already defined in the introduction. We define the periodic white noise W_{per} to be

$$W_{per}(A) = W(A \bmod 1),$$

where $A \bmod 1 = \{(x \bmod 1, y) : (x, y) \in A\}$ and we define $x \bmod 1$ to be the number $x' \in [-\frac{1}{2}, \frac{1}{2})$ such that $x - x'$ is an integer. Now consider cones of the form

$$H(x) := \{(x', y') : |x' - x| < \frac{1}{2}, y > \frac{2}{\pi}\tan|\pi|x' - x||\}.$$

It was noted in [3] that the field $x \mapsto \sqrt{2}W_{per}(H(x))$ has formally the right covariance (6.2), whence a natural sequence of approximation fields $(X_{1,n})$ is obtained by cutting the white noise at the level $1/n$. More precisely we define the truncated cones

$$H_t(x) := H(x) \cap \{(x, y) \in \mathbb{R}^2 : y > e^{-t}\} \tag{6.3}$$

and define the regular field $X_{1,n}$ by the formula

$$X_{1,n}(x) := \sqrt{2}W_{per}(H_{\log n}(x)). \tag{6.4}$$

[2]Observe that we have in fact multiplied the standard definition by $\sqrt{2}$ to get the critical field. Also the innocent constant term $2\sqrt{\log 2}G$ is often omitted in the definition.

The third approximation fields $X_{3,n}$ are defined by using a Hölder continuous function $\varphi \in L^1(\mathbb{R})$ that satisfies $\int \varphi = 1$ and possesses the decay

$$|\varphi(x)| \leq \frac{C}{(1+|x|)^{1+\delta}}$$

for some $C, \delta > 0$. We then set $X_{3,n} := \varphi_{1/n} * X_{per}$, where $X_{per}(x) = X(x + 2\pi\mathbb{Z})$ is the natural lift of X to a map $\mathbb{R} \to \mathbb{R}$. This form of convolution is fairly general, and encompasses convolutions against functions $\widetilde{\varphi}$ defined on the circle whose support do not contain the point $(-1, 0)$.

Example 6.1. Let u be the harmonic extension of X in the unit disc and consider the approximating fields $X_n(x) = u(r_n x)$ for $x \in S^1$ and for an increasing sequence of radii r_n tending to 1. Then $X_n(x)$ is obtained from X by taking a convolution against the Poisson kernel φ_{ε_n} on the real axis, where $\varphi(x) = \frac{2}{1+4\pi^2 x^2}$ and $\varepsilon_n = \log \frac{1}{r_n}$. This kind of approximations might be useful for example in studying fields that have been considered in [24].

The fourth approximation fields $X_{4,n}$ are defined by using a wavelet $\psi\colon \mathbb{R} \to \mathbb{R}$, that is obtained from a multiresolutional analysis, see [32, Definition 2.2]. We further assume that ψ is of bounded variation, so that the distributional derivative ψ' is a finite measure. Finally we require the mild decay

$$|\psi(x)| \leq C(1+|x|)^{-\alpha} \tag{6.5}$$

with some constants $C > 0$ and $\alpha > 2$, and the tail condition

$$\int_{-\infty}^{\infty} (1+|x|) d|\psi'|(x) < \infty. \tag{6.6}$$

Remark 6.2. The conditions (6.5) and (6.6) are fairly general, especially the standard Haar wavelets satisfy them.

With the above definitions it follows from [32, Proposition 2.21] that the periodized wavelets

$$\psi_{j,k}(x) := 2^{j/2} \sum_{l=-\infty}^{\infty} \psi(2^j(x-l)-k)$$

together with the constant function 1 form a basis for the space $L^2([0,1])$.

We next consider vaguelets that can be thought of as half-integrals of wavelets. Our presentation will be rather succinct – another more detailed account can be found in the article by Tecu [30]. The vaguelet $\nu\colon \mathbb{R} \to \mathbb{R}$ is constructed by setting

$$\nu(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\psi(t)}{\sqrt{|x-t|}}\, dt. \tag{6.7}$$

An easy computation utilizing the decay of ψ and the fact that $\int \psi = 0$ verifies that $\nu\colon \mathbb{R} \to \mathbb{R}$ satisfies

$$|\nu(x)| \leq \frac{C}{(1+|x|)^{1+\delta}} \tag{6.8}$$

for some $C, \delta > 0$. We may then define the periodized functions

$$\nu_{j,k}(x) := \sum_{l \in \mathbb{Z}} \nu(2^j(x-l)-k) \tag{6.9}$$

for all $j \geq 0$ and $0 \leq k \leq 2^j - 1$. It is straightforward to check that the Fourier coefficients of $\nu_{j,k}$ satisfy

$$\widehat{\nu}_{j,k}(n) = \frac{\widehat{\psi}_{j,k}(n)}{\sqrt{|2\pi n|}} \quad \text{when } n \neq 0.$$

The field $X_{4,n}$ can now be defined by

$$X_{4,n}(x) := 2\sqrt{\log 2}G + \sqrt{2\pi}\sum_{j=0}^{n}\sum_{k=0}^{2^j-1} A_{j,k}\nu_{j,k}(x), \tag{6.10}$$

where G and $A_{j,k}$ are independent standard Gaussian random variables. To see that this indeed has the right covariance one may first notice that

$$Y = \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1} A_{j,k}\psi_{j,k}(x)$$

defines a distribution valued field satisfying $\mathbb{E}\,\langle Y,u\rangle\langle Y,v\rangle = \langle u,v\rangle$ for all 1-periodic C^∞ functions u and v. The field $X_{4,n}(x)$ is essentially the half integral of this field, whose covariance is given by

$$\mathbb{E}\,\langle I^{1/2}Y,u\rangle\langle I^{1/2}Y,v\rangle = \mathbb{E}\,\langle Y,I^{1/2}u\rangle\langle Y,I^{1/2}v\rangle = \langle I^{1/2}u,I^{1/2}v\rangle = \langle Iu,v\rangle,$$

where the lift semigroup $I^\beta f$ for functions f on S^1 is defined by describing its action on the Fourier basis: $I^\beta e^{2\pi inx} = (2\pi|n|)^{-\beta}e^{2\pi inx}$ for any $n \neq 0$ and $I^\beta 1 = 0$. A short calculation shows that the operator I has the right integral kernel $\frac{1}{\pi}\log\frac{1}{2|\sin(\pi(x-y))|}$.

Proof of Theorem 1.3. The road map for the proof (as well as for the rest of the section) is as follows:

1. We first show in Lemma 6.4 below that the chaos measures constructed from the white noise approximations converge weakly in L^p by comparing it to the exactly scale invariant field on the unit interval by using Proposition A.2.

2. Next we verify in Lemma 6.5 that the Fourier series approximations give the same result as the white noise approximations. This is done by a direct comparison of their covariances to verify the assumptions of Theorem 1.1.

3. Thirdly we deduce in Lemma 6.7 that convolution approximations also yield the same result by comparing a convolution against a Gaussian kernel to the Fourier series and again using Theorem 1.1.

4. Fourthly we prove in Lemma 6.8 that a vaguelet approximation yields the same result by comparing it against the white noise approximation.

5. Finally, in Lemma 6.9 convergence in probability is established for the Fourier series, convolution and vaguelet approximations by invoking Theorem 4.4.

After the steps (1)–(5) the proof of Theorem 1.3 is complete.

The following lemma gives a quantitative estimate that can be used to compare fields defined using the hyperbolic white noise on $\mathbb{H}$.

Lemma 6.3. *Let U be an open subset of $\{(x,y) \in \mathbb{H} : y < 1\}$ such that the set $\{(x,y) \in U : y = s\}$ is an interval for all $0 < s < 1$. Let $f(s)$ denote the length of this interval and assume that $f(s) \leq Cs^{1+\delta}$ for some $\delta > 0$. Then the map $(x,s) \mapsto W(U_s + x)$ admits a modification that is almost surely continuous in $[a,b] \times [0,1]$ for any $a < b$, and almost surely the maps $x \mapsto W(U_s + x)$ tend to $W(U + x)$ uniformly when $s \to 0$. Here $U_s = \{(x,y) \in U : y > s\}$.*

Proof. Let us first show that

$$\mathbb{E}\,|W(U_s+x)-W(U_s+y)|^2 \leq \widetilde{C}|x-y|^{\frac{\delta}{1+\delta}}.$$

for some $\widetilde{C}>0$. By translation invariance of the covariance it is enough to consider $\mathbb{E}\,|W(U_s+x)-W(U_s)|^2$ and we can clearly assume that $0<x<1$. Obviously the 1-dimensional Lebesgue measure of the set $((U_s+x)\cap\{y=a\})\Delta(U_s\cap\{y=a\})$ equals $2\min(f(a),x)$. Hence we have

$$\begin{aligned}\mathbb{E}\,|W(U_s+x)-W(U_s)|^2 &= 2\int_s^1 \frac{\min(f(y),x)}{y^2}\,dy \leq 2\max(1,C)\int_0^1 \frac{\min(y^{1+\delta},x)}{y^2}\,dy\\ &= 2\max(1,C)\Big((1+\delta^{-1})x^{\frac{\delta}{1+\delta}}-x\Big) \leq \widetilde{C}x^{\frac{\delta}{1+\delta}}.\end{aligned}$$

Notice next that

$$\mathbb{E}\,|W(U_s)-W(U_t)|^2 = \int_s^t \frac{f(u)}{u^2}\,du \leq \frac{C}{\delta}(t^\delta - s^\delta).$$

It follows that the map $(x,s)\mapsto W(U_s+x)$ is Hölder-regular both in x and s, and therefore also jointly. By Lemma 2.2 the realizations can be chosen to be almost surely continuous in the rectangle $[a,b]\times[0,1]$ which obviously yields the claim. □

The claim concerning the approximating fields $X_{1,n}$ follows from the next lemma by taking into account the definitions (6.3) and (6.4). In the proof we identify the field on the unit circle locally as a perturbation of the exactly scaling field on the unit interval. For the chaos corresponding to the last mentioned field the fundamental result on convergence was proven in [12], and we use this fact as the basis of the proof of the following lemma.

Lemma 6.4. *Let either $\beta<1$ and ρ_t be the Lebesgue measure on the circle, or let $\beta=1$ and $d\rho_t(x)=\sqrt{t}\,dx$. Then the measures*

$$e^{\beta\sqrt{2}W_{per}(H_t(x))-\beta^2\mathbb{E}\,[W_{per}(H_t(x))^2]}\,d\rho_t(x)$$

defined on the unit circle (which we identify with $\mathbb{R}/\mathbb{Z}$) converge weakly in $L^p(\Omega)$ to a non-trivial measure μ_{β,S^1} for $0<p<1$.

Proof. As our starting point we know that the measures defined by

$$d\widetilde{\mu}_t(x) := e^{\beta\sqrt{2}W(A_t(x))-\beta^2\mathbb{E}\,[W(A_t(x))^2]}\,d\rho_t(x)$$

on the interval $[-\frac12,\frac12]$ converge weakly in $L^p(\Omega)$ to a non-trivial measure for $0<p<1$ under the assumptions we have on β and ρ_t. Here A_t stands for the cone defined in (1.2) in the introduction. One should keep in mind that we are using the same hyperbolic white noise when defining both W and W_{per}.

Let us split the cones H_t into two sets H_t^+ and H_t^-, where

$$H_t^+(x) := H_t(x)\cap\{(x,y)\in\mathbb{H} : y\geq 1\} \text{ and } H_t^-(x) := H_t(x)\cap\{(x,y)\in\mathbb{H} : y<1\}.$$

Clearly $W_{per}(H_t(x)) = W_{per}(H_t^+(x)) + W_{per}(H_t^-(x))$ and by elementary geometry it is easy to see that if we restrict x to the interval $(-\delta_0,\delta_0)$ where $\delta_0=\frac12-\frac{\arctan(\pi/2)}{\pi}\approx 0.18$, we have $(W_{per}(H_t^-(x)))_{x\in(-\delta_0,\delta_0)} = (W(H_t^-(x)))_{x\in(-\delta_0,\delta_0)}$. Hence our aim is to first verify the convergence on the interval $(-\delta_0,\delta_0)$.

Write then

$$\begin{aligned} Y_t(x) &= W_{per}(H_t(x)), \\ Y_t^+(x) &= W_{per}(H_t^+(x)), \\ Y_t^-(x) &= W(H_t^-(x)) \end{aligned}$$

and similarly for the limit fields (which clearly exist in the sense of distributions) write

$$\begin{aligned} Y(x) &= W_{per}(H(x)), \\ Y^+(x) &= W_{per}(H^+(x)), \\ Y^-(x) &= W(H^-(x)). \end{aligned}$$

Let $X_t(x) := W(A_t(x))$ and $X(x) := W(A(x))$ and define $Z_t(x) := Y_t^-(x) - X_t(x)$ so that we may write $Y_t^-(x) = X_t(x) + Z_t(x)$. We next make sure that $Z_t(x)$ is a Hölder regular field, the realizations of which converge almost surely uniformly to the Hölder regular Gaussian field $Z(x) := Y^-(x) - X(x)$.

The field $Z(x)$ decomposes into a sum $L(x) + R(x) + T(x)$, where $L(x) = -W(\widetilde{L} + x)$, $R(x) = -W(\widetilde{R} + x)$ and $T(x) = -W(\widetilde{T} + x)$ with

$$\begin{aligned} \widetilde{R} &= \{(x,y) : \frac{1}{\pi}\arctan(\frac{\pi}{2}y) < x \le \frac{y}{2}, y < 1\} \\ \widetilde{L} &= \{(-x,y) : (x,y) \in \widetilde{R}\} \\ \widetilde{T} &= \{(x,y) : -\frac{1}{2} \le x \le \frac{1}{2}, y \ge 1\}. \end{aligned}$$

We define the truncated versions of L_t, R_t and T_t by cutting the respective sets at the level e^{-t} as usual, so that $Z_t(x) = L_t(x) + R_t(x) + T_t(x)$. Clearly $T_t(x) = T(x)$ for $t \ge 0$.

Let now $f(u) = \frac{u}{2} - \frac{1}{\pi}\arctan(\frac{\pi}{2}u)$. Using the Taylor series of $\arctan(u) = u - \frac{u^3}{3} + \frac{u^5}{5} - \frac{u^7}{7} + \dots$ we have

$$f(u) = \frac{\pi^2}{24}u^3 + O(u^5),$$

so $f(u) \le Cu^3$ for some constant $C > 0$. It follows from Lemma 6.3 that $L_t(x)$ and $R_t(x)$ converge almost surely uniformly to the fields $L(x)$ and $R(x)$, so $Z_t(x)$ converges almost surely uniformly to $Z(x)$ as $t \to \infty$.

Note that $\mathbb{E}\left[Z_t(x)X_t(x)\right]$ tends to a finite constant as $t \to \infty$, so the assumptions of Proposition A.2 are satisfied. Therefore the measures

$$\nu_t = \int f(x) e^{\beta\sqrt{2}Y_t^-(x) - \beta^2\mathbb{E}\left[Y_t^-(x)^2\right]}\, d\rho_t(x)$$

on $(-\delta, \delta)$ converge weakly in $L^p(\Omega)$ for all $0 < p < 1$. Because Y^+ is a regular field, we may again use Proposition A.2 to conclude that also the measures

$$\widetilde{\mu}_t(f) = \int f(x) e^{\beta\sqrt{2}Y_t(x) - \beta^2\mathbb{E}\left[Y_t(x)^2\right]}\, d\rho_t(x)$$

on $(-\delta, \delta)$ converge in $L^p(\Omega)$. By the translation invariance of the field the same holds for any interval of length 2δ. Let $I_1, \dots, I_n$ be intervals of length 2δ that cover the unit circle and let $p_1, \dots, p_n \in C(S^1)$ be a partition of unity with respect to the cover I_k. The measure

$$\mu_t(f) = \int f(x) e^{\beta\sqrt{2}Y_t(x) - \beta^2\mathbb{E}\left[Y_t(x)^2\right]}\, d\rho_t(x)$$

on the whole unit circle can be expressed as a sum $d\mu_t(x) = p_1(x)d\widetilde{\mu}_t^{(1)}(x) + \cdots + p_2(x)d\widetilde{\mu}_t^{(n)}(x)$. Because each of the summands converges in $L^p(\Omega)$, we see that also the family of measures μ_t converges in $L^p(\Omega)$. $\square$

Lemma 6.5. *Let either $\beta < 1$ and $d\rho_n(x) = dx$ for all $n \geq 1$ or let $\beta = 1$ and $d\rho_n(x) = \sqrt{\log n}\, dx$. Then the measures*

$$d\mu_{2,n}(x) := e^{\beta X_{2,n}(x) - \frac{\beta^2}{2}\mathbb{E}\,[X_{2,n}(x)^2]}\, d\rho_n(x)$$

converge in distribution to the random measure μ_{β,S^1} constructed in Lemma 6.4.

Proof. Let $f_n(x) := \mathbb{E}\,[X_{2,n}(x)X_{2,n}(0)]$. It is straightforward to calculate that

$$f_n(x) = 4\log 2 + 2\sum_{k=1}^{n} \frac{\cos(2\pi kx)}{k}.$$

In particular $f_n(0) = 4\log 2 + 2H_n$, where H_n is the nth Harmonic number, $H_n = \log n + \gamma + O(\frac{1}{n})$ with γ being the Euler–Mascheroni constant. Let $f(x) := 4\log 2 + 2\log\frac{1}{2|\sin(\pi x)|}$ be the limit covariance and define $g_n(x) := f(x) - f_n(x)$. One can easily compute that for $0 < x \leq \frac{1}{2}$ we have

$$g_n'(x) = -\frac{2\pi\cos(2\pi(n+\frac{1}{2})x)}{\sin(\pi x)}.$$

In particular the maximums and minimums of the difference $g_n(x)$ occur at the points $x_j^{(n)} = \frac{2j+1}{4n+2}$, $0 \leq j \leq n$. Consider the telescoping sum

$$g_n(x_j^{(n)}) = (g_n(x_j^{(n)}) - g_n(x_{j+1}^{(n)})) + \cdots + (g_n(x_{n-1}^{(n)}) - g_n(x_n^{(n)})) + g_n(x_n^{(n)}). \tag{6.11}$$

Here the terms in parentheses form an alternating series whose terms are decreasing in absolute value. Moreover, the term $g_n(x_0^{(n)}) - g_n(x_1^{(n)})$ stays bounded as $n \to \infty$ and the term $g_n(x_n^{(n)})$ goes to 0 as $n \to \infty$. All this is obvious from writing

$$\begin{aligned}
g_n(x_{j+1}^{(n)}) - g_n(x_j^{(n)}) &= \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} g_n'(t)\,dt = -2\pi\int_{\frac{2j+1}{4n+2}}^{\frac{2j+3}{4n+2}} \frac{\cos(\pi(2n+1)t)}{\sin(\pi t)}\,dt \qquad (6.12)\\
&= \frac{-2\pi}{2n+1}\int_{-1/2}^{1/2} \frac{\cos(\pi(y+j+1))}{\sin(\pi\frac{y+j+1}{2n+1})}\,dy\\
&= \frac{(-1)^j 2\pi}{2n+1}\int_{-1/2}^{1/2} \frac{\cos(\pi y)}{\sin(\pi\frac{y+j+1}{2n+1})}\,dy,\\
g_n(x_n^{(n)}) &= -2\log(2) - 2\sum_{k=1}^{n}\frac{(-1)^k}{k}.
\end{aligned}$$

In particular we deduce that

$$\sup_{n\geq 1}\sup_{x \geq x_0^{(n)}} |g_n(x)| < \infty. \tag{6.13}$$

Notice also that for any fixed $\varepsilon > 0$ all the maximums and minimums in the range $x > \varepsilon$ are located at the points $x_j^{(n)}$ with $j > 2\varepsilon n + \varepsilon - \frac{1}{2}$, and

$$\lim_{n\to\infty}\sup_{j > \varepsilon n + \varepsilon - \frac{1}{2}} |g_n(x_{j+1}^{(n)}) - g_n(x_j^{(n)})| = 0$$

by (6.12). From this and (6.11) it follows that the Fourier covariance converges to the limit covariance uniformly in the set $\{|x| > \varepsilon\}$, a fact that could also be deduced from the localized uniform convergence of the Fourier series of smooth functions [34, p. 54, Theorem 6.8].

Consider next the white noise covariance $h_t(x) := 2\mathbb{E}\left[W_{per}(H_t(x))W_{per}(H_t(0))\right]$. By symmetry we may assume all the time that $x > 0$. After a slightly tedious calculation one arrives at the formula

$$h_t(x) = \begin{cases} 4\log 2 + 2\log \frac{1}{2\sin(\pi x)}, & \text{if } x > \frac{2}{\pi}\arctan(\frac{\pi}{2}e^{-t}) \\ -2xe^t + 2t - 2\log(\cos(\frac{\pi}{2}x)) + \log(\pi^2 e^{-2t} + 4) & \\ + \frac{2\arctan(\frac{\pi}{2}e^{-t})}{\frac{\pi}{2}e^{-t}} - 2\log(\pi), & \text{if } x \le \frac{2}{\pi}\arctan\left(\frac{\pi}{2}e^{-t}\right). \end{cases}$$

Let us consider the approximation along the sequence $t_n = \log(n)$. Then $h_{t_n}(0) = 2\log(n) + O(1)$. Moreover at the point $x_n = \frac{2}{\pi}\arctan(\frac{\pi}{2}e^{-t_n}) = \frac{2}{\pi}\arctan(\frac{\pi}{2n})$ we have

$$h_{t_n}(x_n) = 4\log 2 + 2\log \frac{1}{2\sin(2\arctan(\frac{\pi}{2n}))} = 2\log(n) + O(1).$$

Because the function h_{t_n} without the bounded term $-2\log(\cos(\frac{\pi}{2}x))$ is linear and decreasing on the interval $[0, x_n]$ we know that it is actually $2\log(n) + O(1)$ on that whole interval. Similarly it is easy to check that for the Fourier series we have $f_n(x) = 2\log(n) + O(1)$ on the interval $[0, x_n]$ because $|f_n'(x)| \le 4\pi n$ and $x_n = O(\frac{1}{n})$. Thus $|f_n(x) - h_{t_n}(x)| = O(1)$ for $x \le x_n$. For $x \ge x_n$ it follows from (6.13) that $|f_n(x) - h_{t_n}(x)| = |g_n(x)|$ is bounded.

From the above considerations and symmetry it follows that the covariances of the fields $X_{1,n}$ and $X_{2,n}$ satisfy the assumptions of Theorem 1.1. This finishes the proof. □

Remark 6.6. The somewhat delicate considerations in the previous proof are necessary because of the fairly unwieldy behaviour of the Dirichlet kernel.

Next we verify that any convolution approximation to the field X also has the same limit.

Lemma 6.7. *Let φ be a Hölder continuous mollifier satisfying $\int_{-\infty}^{\infty}\varphi(x)\,dx = 1$ and $\varphi(x) = O(x^{-1-\delta})$ for some $\delta > 0$. Then the fields $X_{3,n}$ defined on S^1 by using the periodized field on $\mathbb{R}$:*

$$X_{3,n}(x) := (\varphi_{1/n} * X_{per})(x)$$

are Hölder-regular and the measures

$$d\mu_{3,n} := e^{\beta X_{3,n}(x) - \frac{\beta^2}{2}\mathbb{E}\,[X_{3,n}(x)^2]}\,d\rho_n(x),$$

converge in distribution to μ_{β,S^1}. Here ρ_n is the Lebesgue measure if $\beta < 1$ and $d\rho_n = \sqrt{\log n}\,dx$ if $\beta = 1$.

Proof. It is enough to show the assumptions of Theorem 1.1 for one kernel satisfying the conditions of the lemma because of Corollary 5.4, and because of Lemma 6.5 we can do our comparison against the covariance obtained from the Fourier series construction. We will make the convenient choice of $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ as our kernel. The covariance of the field $\varphi_\varepsilon * X_{per}$ is given by $(\psi_\varepsilon * f)(x-y)$, where $\psi_\varepsilon(x) = (\varphi_\varepsilon * \varphi_\varepsilon(-\cdot))(x) = \frac{1}{2\varepsilon\sqrt{\pi}}e^{-\frac{x^2}{4\varepsilon^2}}$ and $f(x) = 4\log 2 + 2\log\frac{1}{2|\sin(\pi x)|}$.

Using the identity $\log\frac{1}{2|\sin(\pi x)|} = \sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k}$, a short computation shows that we can write the difference of the covariances of $X_{2,n}$ (the Fourier field) and $X_{3,n}$ in the form (we may take $y = 0$ as we are in the translation invariant case)

$$2\sum_{k=1}^{n}\frac{\cos(2\pi kx)}{k}(1 - e^{-4\pi^2\frac{k^2}{n^2}}) - 2\sum_{k=n+1}^{\infty}\frac{\cos(2\pi kx)}{k}e^{-4\pi^2\frac{k^2}{n^2}}.$$

Since $1 - e^{-x} \le x$ for $x \ge 0$, the first term is bounded by $2\sum_{k=1}^{n} \frac{4\pi^2 k}{n^2} \le 16\pi^2$. In turn the second term is bounded from above by

$$2\int_n^\infty \frac{e^{-4\pi^2 \frac{t^2}{n^2}}}{t}\, dt = 2\int_1^\infty \frac{e^{-4\pi^2 s^2}}{s}\, ds.$$

Because both of the covariances converge locally uniformly outside the diagonal, we again see that the assumptions of Theorem 1.1 are satisfied. □

Our next goal is to prove the convergence in distribution for the vaguelet approximation $X_{4,n}$. In the lemma below we recall the definition of the field $X_{4,n}$ in (6.10). The elementary bounds on vaguelets we use are gathered in Appendix B.

Lemma 6.8. *Let either $\beta < 1$ and $d\rho_n(x) = dx$ for all $n \ge 1$ or let $\beta = 1$ and $d\rho_n(x) = \sqrt{n\log 2}\, dx$. Then the measures*

$$d\mu_{4,n} := e^{\beta X_{4,n}(x) - \frac{\beta^2}{2}\mathbb{E}\,[X_{4,n}(x)^2]}\, d\rho_n(x)$$

converge in distribution to the random measure μ_{β,S^1} constructed in Lemma 6.4.

Proof. The covariance $C_n(x,y)$ of the field $X_{4,n}$ is given by

$$C_n(x,y) = 4\log 2 + 2\pi \sum_{j=0}^{n} \sum_{k=0}^{2^j-1} \nu_{j,k}(x)\nu_{j,k}(y).$$

Let $\psi_{j,k}$ be the periodized wavelets. Then there exists a constant $D > 0$ such that $\|\psi_{j,k}\|_\infty \le D2^{j/2}$ for all $j \ge 0$, $0 \le k \le 2^j - 1$. It follows from Lemma B.1 and Lemma B.3 that when $|x-y| \le 2^{-n}$, we have

$$\begin{aligned} |C_n(x,x) - C_n(x,y)| &\le 2\pi \sum_{j=0}^{n} \sum_{k=0}^{2^j-1} |\nu_{j,k}(x)||\nu_{j,k}(x) - \nu_{j,k}(y)| \\ &\le 2\pi C\sqrt{|x-y|} \sum_{j=0}^{n} \sum_{k=0}^{2^j-1} |\nu_{j,k}(x)|\|\psi_{j,k}\|_\infty \\ &\le 2\pi ACD\sqrt{|x-y|} \sum_{j=0}^{n} 2^{j/2} \le E. \end{aligned} \tag{6.14}$$

for some constant $E > 0$. From Lemma B.3 it also follows that for any $\varepsilon > 0$ the covariances $C_n(x,y)$ converge uniformly in the set $V_\varepsilon = \{(x,y) : \mathrm{dist}(x,y) \ge \varepsilon\}$. Obviously by definition there is a distributional convergence to the right covariance $4\log 2 + 2\log\frac{1}{2|\sin(\pi(x-y))|}$ and this must agree with the uniform limit in V_ε. Especially, by invoking again the bound from Lemma B.3 we deduce that

$$|C_n(x, x+2^{-n}) - 4\log 2 - 2\log\frac{1}{2\sin(\pi 2^{-n})}| \le 2\pi B. \tag{6.15}$$

Thus by combining (6.14) and (6.15) the covariance satisfies

$$|C_n(x,y) - 2n\log 2| \le F \quad \text{for all } (x,y) \in \{(x,y) : \mathrm{dist}(x,y) \le 2^{-n}\}$$

for some constant $F > 0$. From the known behaviour (see e.g. the end of the proof of Lemma 6.5) of the covariance of the white noise field $X_{1,n}$ it is now easy to see that the assumptions of Theorem 1.1 are satisfied for the pair $(X_{4,n})$ and $(X_{1,n})$. □

Finally we observe that the convergence in lemmas 6.5, 6.7 and 6.8 also takes place weakly in L^p.

Lemma 6.9. *The convergences stated in lemmas 6.5, 6.7 and 6.8 take place in L^p for $0 < p < 1$ (especially in probability).*

Proof. We only prove the claim in the critical case since the subcritical case is similar. We will use the fields $X_{1,n}$ as the fields X_n in Theorem 4.4. Then according to Lemma 6.4 we have that $e^{X_n - \frac{1}{2}\mathbb{E}[X_n^2]}\, d\rho_n$ converges in probability to a measure μ_{1,S^1} when $d\rho_n = \sqrt{\log n}\, dx$.

In the case of the Fourier approximation we can define R_n in Theorem 4.4 to be the nth partial sum of the Fourier series. That is

$$R_n f := \sum_{k=-n}^{n} \widehat{f}(k) e^{2\pi i k x}.$$

Recalling Jackson's theorem on the uniform convergence of Fourier series of Hölder continuous functions, it is straightforward to check that R_n is a linear regularization process.

In the case of convolutions we take R_n to be the convolution against $\frac{1}{\varepsilon_n}\varphi(\frac{x}{\varepsilon_n})$, where $(\varepsilon_n)_{n\geq 1}$ is a sequence of positive numbers tending to 0. The sequence (R_n) obviously satisfies the required conditions.

Finally, we sketch the proof for the vaguelet approximations. This time we employ the sequence of operators

$$R_n f(x) := \int_0^1 f + \sum_{j=0}^{n} \sum_{k=0}^{2^j-1} \left(\int_0^1 \psi_{j,k}(y) \big(I^{-1/2} f(y)\big) dy \right) \nu_{j,k}(x).$$

Because of finiteness of the defining series it is easy to see that (R_n) satisfies the second condition in Definition 4.3. For the first condition we first fix $\alpha \in (0, 1/2)$ and observe that $R_n \nu_{j',k'} = \nu_{j',k'}$ as soon as $n \geq j'$. By the density of vaguelets, in order to verify the first condition it is enough to check that the remainder term tends uniformly to 0 for any $f \in C^\alpha(S^1)$. We begin by noting that $\frac{d}{dx} = -iHI^{-1}$, where H is the Hilbert transform, which yields for $f \in C^\alpha(S^1)$

$$\Big| \int_0^1 \psi_{j,k}(y) \big(I^{-1/2} f(y)\big) \Big| = \Big| \int_0^1 \frac{d}{dy} \psi_{j,k}(y) \big(HI^{+1/2} f(y)\big) \Big| \leq C 2^{-\alpha j}, \qquad x \in [0,1),$$

since $HI^{+1/2} f(x) \in C^{\alpha+1/2}(S^1)$ by the standard mapping properties of I^β, and the Hilbert transform is bounded on any of the C^α-spaces. Above, the final estimate was obtained by computing for any $g \in C^{\alpha+1/2}(S^1)$ with periodic continuation G to $\mathbb{R}$ that

$$\begin{aligned} \left| \int_0^1 \frac{d}{dx} \psi_{j,0}(x) g(x) \right| &= \left| \int_{-\infty}^{\infty} 2^{\frac{3}{2}j} d\psi'(2^j x) G(x) \right| = 2^{j/2} \left| \int_{-\infty}^{\infty} d\psi'(x) (G(2^{-j}x) - G(0)) \right| \\ &\leq 2^{j/2} \int_{-\infty}^{\infty} |d\psi'(x)| (2^{-j}x)^{\alpha+1/2} \leq 2^{-\alpha j} \int_{-\infty}^{\infty} |d\psi'(x)| (1 + |x|). \end{aligned}$$

The last integral is finite by the assumption (6.6). Together with Lemma B.3 this obviously yields the desired uniform convergence.

The proofs of the lemmas 6.5, 6.7 and 6.8 show that the covariances stay at a bounded distance from the covariance of the field $X_{1,n}$, and therefore a standard application of Kahane's convexity inequality gives us an L^p bound. Combining this with Theorem 4.4 yields the result. □

As noted in the beginning of this section, having proved all the lemmas above we may conclude the proof of Theorem 1.3. □

Remark 6.10. In the case of vaguelet approximations we may also rewrite

$$X(x) = \sum_{i=1}^{\infty} \widetilde{A}_i \widetilde{\nu}_i(x),$$

where $\widetilde{A}_i$ and $\widetilde{\nu}_i$ are the random coefficents and vaguelets appearing in (6.10) ordered in their natural order. The convergence and uniqueness then also holds for the chaos constructed from the fields

$$\widetilde{X}_{4,n} := \sum_{i=1}^{n} \widetilde{A}_i \widetilde{\nu}_i(x),$$

with the normalizing measure $d\rho_n(x) = \sqrt{\log n}\, dx$.

Remark 6.11. There are many interesting questions that we did not touch in this paper. For example (this question is due to Vincent Vargas), it is natural to ask whether the convergence or uniqueness of the derivative martingale [10] depends on the approximations used.

A Localization

The Proposition A.2 below is needed in a localization procedure in Lemma 6.4 that is used to carry results from the real line to the unit circle. For its proof we need the following lemma.

Lemma A.1. *Assume that μ_n is a sequence of random measures that converges to μ weakly in $L^p(\Omega)$. Let $F\colon \Omega \to C(\mathcal{T})$ be a function valued random variable and assume that there exists $q > 0$ such that*

$$\mathbb{E} \left| \sup_{x \in \mathcal{T}} F(x) \right|^{\alpha} < \infty$$

for some $\alpha > \frac{pq}{p-q}$. Then $\int F(x)\, d\mu_n(x)$ tends to $\int F(x)\, d\mu(x)$ in $L^q(\Omega)$.

Proof. It is again enough to show that any subsequence possesses a converging subsequence with the right limit. To simplify notation let us denote by μ_n an arbitrary subsequence of the original sequence.

Directly from the definition of the metric in the space $\mathcal{M}^+$ we see that $\mu_n \to \mu$ in probability, meaning that we can pick a subsequence μ_{n_j} that converges almost surely. Then the almost sure convergence holds also for the sequence $\int F(x)\, d\mu_{n_j}(x)$. Finally, for any allowed value of q a standard application of Hölder's inequality shows that $\mathbb{E} |\int F(x)\, d\mu_{n_j}(x)|^{q+\varepsilon}$ is uniformly bounded for some $\varepsilon > 0$. This yields uniform integrability and we may conclude. □

Proposition A.2. *Let (X_n) and (Z_n) be two sequences of (jointly Gaussian) Hölder-regular Gaussian fields on $\mathcal{T}$. Assume that the pseudometrics arising in Definition 2.1 can be chosen to have the same Hölder exponent and constant for all the fields Z_n. Assume further that there exists a Hölder-regular Gaussian field Z such that Z_n converges to Z uniformly almost surely and that $\mathbb{E}\,[X_n(x)Z_n(x)]$ converges uniformly to some bounded continuous function $x \mapsto \mathbb{E}\,[X(x)Z(x)]$. Then if the measures*

$$d\mu_n(x) := e^{X_n(x) - \frac{1}{2}\mathbb{E}\,[X_n(x)^2]}\, d\rho_n(x)$$

converge weakly in $L^p(\Omega)$ to a measure μ, also the measures

$$\begin{aligned} d\nu_n(x) &:= e^{(X_n(x)+Z_n(x))-\frac{1}{2}\mathbb{E}\,[(X_n(x)+Z_n(x))^2]}\,d\rho_n(x) \\ &= e^{Z_n(x)-\frac{1}{2}\mathbb{E}\,[Z_n(x)^2]-\mathbb{E}\,[X_n(x)Z_n(x)]}\,d\mu_n(x) \end{aligned}$$

converge weakly in $L^q(\Omega)$ for all $q < p$ to the measure

$$d\nu(x) := e^{Z(x)-\frac{1}{2}\mathbb{E}\,[Z(x)^2]-\mathbb{E}\,[X(x)Z(x)]}\,d\mu(x).$$

Proof. By a standard application of the Borell–TIS inequality [1, Theorem 4.1.2] we have the following uniform bound

$$\mathbb{E}\,e^{r\sup_{x\in\mathcal{T}} Z_n(x)} \le C_r \tag{A.1}$$

for all $r > 0$. Fix $\varepsilon > 0$ and for all $n \ge 1$ define

$$A_n^\varepsilon := \{\omega \in \Omega : \sup_{x\in\mathcal{T}} |Z_k(x) - Z(x)| < \varepsilon \text{ for all } k \ge n\}.$$

By the assumption on uniform convergence we have $\mathbb{P}[A_n^\varepsilon] \to 1$ as $n \to \infty$.

Fix $f \in C(\mathcal{T})$, which we may assume to be non-negative, and let $0 < q < p$. We first show that

$$\mathbb{E}\,[\chi_{\Omega\setminus A_n^\varepsilon}|\nu_n(f) - \nu(f)|^q] \to 0$$

as $n \to \infty$. It is enough to verify uniform integrability by checking that

$$\sup_{n\ge1} \mathbb{E}\,|\nu_n(f)|^{p'} + \mathbb{E}\,|\nu(f)|^{p'} < \infty \tag{A.2}$$

for some $q < p' < p$. This in turn follows easily from the assumed uniform L^p bound for μ_n by using Hölder's inequality together with (A.1).

To handle the remaining term $\mathbb{E}\,[\chi_{A_n^\varepsilon}|\nu_n(f) - \nu(f)|^q]$ we use the defining property of the set A_n^ε, i.e. $|Z_n(x) - Z(x)| < \varepsilon$ for all $x \in \mathcal{T}$. By choosing n large enough and by using (A.1) we may further assume that $\sup_{x\in\mathcal{T}} |\mathbb{E}\,[Z_n(x)^2] - \mathbb{E}\,[Z(x)^2]| < \varepsilon$ and $\sup_{x\in\mathcal{T}} |\mathbb{E}\,[Z_n(x)X_n(x)] - \mathbb{E}\,[Z(x)X(x)]| < \varepsilon$. It follows that when $\omega \in A_n^\varepsilon$, we have

$$e^{-3\varepsilon}c_n(f) \le \nu_n(f) \le e^{3\varepsilon}c_n(f),$$

where

$$c_n(f) = \int f(x)e^{Z(x)-\frac{1}{2}\mathbb{E}\,[Z(x)^2]-\mathbb{E}\,[Z(x)X(x)]}\,d\mu_n(x).$$

By combining this with the bound (A.2) we see that $\mathbb{E}\,|\nu_n(f) - c_n(f)|^q \to 0$ as $\varepsilon \to 0$, uniformly in n. Finally, by Lemma A.1 we have $c_n(f) \to \nu(f)$ in $L^q(\Omega)$. This finishes the proof. □

B Estimates for vaguelets

In this appendix we have collected a couple of elementary estimates concerning vaguelets, see (6.9) in Section 6 for the definition of $\nu_{j,k}$.

Lemma B.1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bounded integrable function and let*

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{f(t)}{\sqrt{|x-t|}}\,dt$$

be its half-integral. Then there exists a constant $C > 0$ (not depending on f) such that for all $x, y \in \mathbb{R}$ we have

$$|F(x) - F(y)| \le C\|f\|_\infty \sqrt{|x-y|}.$$

Proof. Clearly it is enough to show that

$$\int_{-\infty}^{\infty} \left| \frac{1}{\sqrt{|x-t|}} - \frac{1}{\sqrt{|y-t|}} \right| dt \le C\sqrt{|x-y|}.$$

Notice that the integrand can be approximated by

$$\begin{aligned}\left| \frac{1}{\sqrt{|x-t|}} - \frac{1}{\sqrt{|y-t|}} \right| &= \frac{\big||y-t| - |x-t|\big|}{|x-t|\sqrt{|y-t|} + \sqrt{|x-t|}|y-t|} \\ &\le \frac{|x-y|}{|x-t|\sqrt{|y-t|} + \sqrt{|x-t|}|y-t|}.\end{aligned}$$

We can without loss of generality assume that $x < y$ and split the domain of integration to the intervals $(-\infty, x]$, $[x, \frac{x+y}{2}]$, $[\frac{x+y}{2}, y]$ and $[y, \infty)$. On each of the intervals the value of the integral is easily estimated to be less than some constant times $\sqrt{|x-y|}$, which gives the result. □

Lemma B.2. *We have*

$$\nu_{j,0}(x) \le \frac{c}{(1 + 2^j \operatorname{dist}(x,0))^{1+\delta}}$$

for some constant $c > 0$. Here $\operatorname{dist}(x,y) = \min\{|x-y+k| : k \in \mathbb{Z}\}$.

Proof. Without loss of generality we may assume that $0 \le x < 1$ and let $d = \operatorname{dist}(x,0)$. We have

$$\begin{aligned}|\nu_{j,0}(x)| \le& \sum_{l\in\mathbb{Z}} \frac{C}{(1+2^j|x-l|)^{1+\delta}} \le \sum_{l=0}^{\infty} \frac{C}{(1+2^jx+2^jl)^{1+\delta}} + \sum_{l=1}^{\infty} \frac{C}{(1+2^jl-2^jx)^{1+\delta}} \\ \le& \frac{2C}{(1+2^jd)^{1+\delta}} + 2\sum_{l=1}^{\infty} \frac{C}{(1+2^jl)^{1+\delta}} \\ \le& \frac{2C}{(1+2^jd)^{1+\delta}} + \frac{2C}{(1+2^j)^{1+\delta}} + 2\int_1^{\infty} \frac{C}{(1+2^ju)^{1+\delta}}\, du \\ \le& 4C\frac{1}{(1+2^jd)^{1+\delta}} + \frac{2C}{\delta}\frac{1}{2^j(1+2^j)^{\delta}} \le \frac{c}{(1+2^jd)^{1+\delta}}.\end{aligned}$$ □

Lemma B.3. *There exists a constant $A > 0$ such that*

$$\sum_{k=0}^{2^j-1} |\nu_{j,k}(x)| \le A$$

for all $j \ge 0$ and $x \in \mathbb{R}$.

Moreover, there exists a constant $B > 0$ such that for all $n \ge 0$ and $x, y \in \mathbb{R}$ satisfying $\operatorname{dist}(x,y) \ge 2^{-n}$ we have

$$\sum_{j=n}^{\infty} \sum_{k=0}^{2^j-1} |\nu_{j,k}(x)\nu_{j,k}(y)| \le B.$$

Proof. By using Lemma B.2 and the fact that $\nu_{j,k}(x) = \nu_{j,0}(x - k2^{-j})$ we have

$$\begin{aligned}\sum_{k=0}^{2^j-1} |\nu_{j,k}(x)| = \sum_{k=0}^{2^j-1} |\nu_{j,0}(x-k2^{-j})| \le& \sum_{k=0}^{2^j-1} \frac{c}{(1+2^j\operatorname{dist}(x-k2^{-j},0))^{1+\delta}} \\ \le& 2c\sum_{k=0}^{\infty} \frac{1}{(1+k)^{1+\delta}} < \infty,\end{aligned}$$

which shows the first claim.

Again invoking Lemma B.2 and the fact that $\nu_{j,k}(x) = \nu_{j,0}(x - k2^{-j})$ we may estimate

$$\begin{aligned}
&\sum_{j=n}^{\infty}\sum_{k=0}^{2^j-1} |\nu_{j,k}(x)\nu_{j,k}(y)| \\
\leq &\sum_{j=n}^{\infty}\sum_{k=0}^{2^j-1} \frac{c^2}{(1+2^j\operatorname{dist}(x-k2^{-j},0))^{1+\delta}(1+2^j\operatorname{dist}(y-k2^{-j},0))^{1+\delta}} \\
\leq &\sum_{j=n}^{\infty}\sum_{k=0}^{2^j-1} \frac{2c^2}{\max((1+k)^{1+\delta},(1+2^{j-n-1})^{1+\delta})} \\
\leq &2c^2\sum_{j=n}^{\infty}\Big(\frac{1+2^{j-n-1}}{(1+2^{j-n-1})^{1+\delta}} + \sum_{k=2^{j-n-1}+1}^{\infty}\frac{1}{(1+k)^{1+\delta}}\Big) \\
\leq &2c^2\sum_{j=n}^{\infty}\Big(\frac{1}{(1+2^{j-n-1})^{\delta}} + \int_{2^{j-n-1}}^{\infty}\frac{1}{(1+x)^{1+\delta}}\,dx\Big) \\
\leq &2c^2(1+\frac{1}{\delta})\sum_{j=n}^{\infty}\frac{1}{(1+2^{j-n-1})^{\delta}} \leq 2c^2(1+\frac{1}{\delta})\sum_{j=0}^{\infty}2^{-\delta(j-1)} = B < \infty,
\end{aligned}$$

giving us the second claim. □

References

[1] Adler R. J. and Taylor J. E., *Random fields and geometry*, Springer Monographs in Mathematics, Springer, New York, 2007. MR-2319516

[2] Aidekon E. and Shi Z., *The Seneta-Heyde scaling for the branching random walk*, Ann. Probab. **42** (2014), no. 3, 959–993. MR-3189063

[3] Astala K., Jones P., Kupiainen A., and Saksman E., *Random conformal weldings*, Acta Math. **207** (2011), no. 2, 203–254. MR-2892610

[4] Bacry E. and Muzy J. F., *Log-infinitely divisible multifractal processes*, Comm. Math. Phys. **236** (2003), no. 3, 449–475. MR-2021198

[5] Berestycki N., *An elementary approach to Gaussian multiplicative chaos.*, arXiv:1506.09113 (2015).

[6] Berestycki N., Sheffield S., and Sun X., *Equivalence of Liouville measure and Gaussian free field*, arXiv:1410.5407v3 (2015).

[7] Bogachev V. I., *Measure theory. Vol. I, II*, Springer-Verlag, Berlin, 2007. MR-2267655

[8] Bramson M., *Convergence of solutions of the Kolmogorov equation to travelling waves*, Mem. Amer. Math. Soc. **44** (1983), no. 285, iv+190. MR-705746

[9] David F., Rhodes R., and Vargas V., *Liouville quantum gravity on complex tori*, J. Math. Phys. **57** (2016), no. 2, 022302, 25. MR-3450564

[10] Duplantier B., Rhodes R., Sheffield S., and Vargas V., *Critical Gaussian multiplicative chaos: convergence of the derivative martingale*, Ann. Probab. **42** (2014), no. 5, 1769–1808. MR-3262492

[11] Duplantier B., Rhodes R., Sheffield S., and Vargas V., *Log-correlated Gaussian fields: an overview*, arXiv:1407.5605 (2014).

[12] Duplantier B., Rhodes R., Sheffield S., and Vargas V., *Renormalization of critical Gaussian multiplicative chaos and KPZ relation*, Comm. Math. Phys. **330** (2014), no. 1, 283–330. MR-3215583

[13] Duplantier B. and Sheffield S., *Duality and the Knizhnik-Polyakov-Zamolodchikov relation in Liouville quantum gravity*, Phys. Rev. Lett. **102** (2009), no. 15, 150603, 4. MR-2501276

[14] Duplantier B. and Sheffield S., *Liouville quantum gravity and KPZ*, Invent. Math. **185** (2011), no. 2, 333–393. MR-2819163

[15] Duplantier B. and Sheffield S., *Schramm-Loewner evolution and Liouville quantum gravity*, Phys. Rev. Lett. **107** (2011), no. 13, 131305.

[16] Fyodorov Y. V. and Keating J. P., *Freezing transitions and extreme values: random matrix theory, and disordered landscapes*, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **372** (2014), no. 2007, 20120503, 32. MR-3151088

[17] Grafakos L., *Modern Fourier analysis*, second ed., Graduate Texts in Mathematics, vol. 250, Springer, New York, 2009. MR-2463316

[18] Høegh-Krohn R., *A general class of quantum fields without cut-offs in two space-time dimensions*, Comm. Math. Phys. **21** (1971), 244–255. MR-0292433

[19] Huang Y., Rhodes R., and Vargas V., *Liouville Quantum Gravity on the unit disk*, arXiv:1502.04343 (2015).

[20] Kahane J.-P., *Sur le chaos multiplicatif*, Ann. Sci. Math. Québec **9** (1985), no. 2, 105–150. MR-829798

[21] Kahane J.-P., *Positive martingales and random measures*, Chinese Ann. Math. Ser. B **8** (1987), no. 1, 1–12, A Chinese summary appears in Chinese Ann. Math. Ser. A **8** (1987), no. 1, 136. MR-886744

[22] Kallenberg O., *Foundations of modern probability*, second ed., Probability and its Applications (New York), Springer-Verlag, New York, 2002. MR-1876169

[23] Kupiainen A., Rhodes R., and Vargas V., *Conformal Ward and BPZ identities for Liouville quantum field theory*, arXiv:1512.01802 (2015).

[24] Miller J. and Sheffield S., *Quantum Loewner evolution*, Duke Math. J. **165** (2016), no. 17, 3241–3378. MR-3572845

[25] Rhodes R. and Vargas V., *Gaussian multiplicative chaos and applications: a review*, Probab. Surv. **11** (2014), 315–392. MR-3274356

[26] Robert R. and Vargas V., *Gaussian multiplicative chaos revisited*, Ann. Probab. **38** (2010), no. 2, 605–631. MR-2642887

[27] Saksman E. and Webb C., *The Riemann zeta function and Gaussian multiplicative chaos: statistics on the critical line*, arXiv:1609.00027 (2016).

[28] Shamov A., *On Gaussian multiplicative chaos*, J. Funct. Anal. **270** (2016), no. 9, 3224–3261. MR-3475456

[29] Sheffield S., *Conformal weldings of random surfaces: SLE and the quantum gravity zipper*, Ann. Probab. **44** (2016), no. 5, 3474–3545. MR-3551203

[30] Tecu N., *Random Conformal Weldings at criticality*, arXiv:1205.3189 (2012).

[31] Webb C., *Exact asymptotics of the freezing transition of a logarithmically correlated random energy model*, J. Stat. Phys. **145** (2011), no. 6, 1595–1619. MR-2863721

[32] Wojtaszczyk P., *A mathematical introduction to wavelets*, London Mathematical Society Student Texts, vol. 37, Cambridge University Press, Cambridge, 1997. MR-1436437

[33] Yor M., *On Some Exponential Functionals of Brownian Motion*, Exponential Functionals of Brownian Motion and Related Processes, Springer Berlin Heidelberg, Berlin, Heidelberg, 2001, pp. 23–48.

[34] Zygmund A., *Trigonometric series. Vol. I, II*, third ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2002, With a foreword by Robert A. Fefferman. MR-1963498

Acknowledgments. We thank Dario Gasbarra for useful discussions in connection with Lemma 4.2 and Christian Webb for many valuable remarks on the manuscript. Moreover, we are grateful to the referee for the thorough reading of the article and the many insightful comments that helped to improve the paper.

ARTICLE II

J. Junnila

On the Multiplicative Chaos of Non-Gaussian Log-Correlated Fields

International Mathematics Research Notices (2018), *rny196*

II

J. Junnila (2018) "On the Multiplicative Chaos of Non-Gaussian Log-Correlated Fields,"
International Mathematics Research Notices, Vol. 00, No. 0, pp. 1–28
Advance Access publication on Month XX, 2018
doi:10.1093/imrn/rny196

On the Multiplicative Chaos of Non-Gaussian Log-Correlated Fields

Janne Junnila*

Department of Mathematics and Statistics, University of Helsinki, P.O. Box 68, FIN-00014, Finland

**Correspondence to be sent to: e-mail: janne.junnila@helsinki.fi*

We study non-Gaussian log-correlated multiplicative chaos, where the random field is defined as a sum of independent fields that satisfy suitable moment and regularity conditions. The convergence, existence of moments, and analyticity with respect to the inverse temperature are proven for the resulting chaos in the full subcritical range. These results are generalizations of the corresponding theorems for Gaussian multiplicative chaos. A basic example where our results apply is the non-Gaussian Fourier series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}(A_k \cos(2\pi kx) + B_k \sin(2\pi kx)),$$

where A_k and B_k are i.i.d. random variables.

1 Introduction

The theory of multiplicative chaos was originally introduced by Kahane [10, 11] as a continuous analogy of Mandelbrot cascades [14]. Kahane's theory concerns weak*-limits of random measures of the form

$$\mathrm{d}\mu_n(x) = e^{\sum_{k=1}^n X_k(x) - \frac{1}{2}\sum_{k=1}^n \mathbb{E} X_k(x)^2}\, \mathrm{d}\lambda(x),$$

Received December 15, 2017; Revised May 8, 2018; Accepted July 26, 2018
Communicated by Prof. Scott Sheffield

where X_k are independent centered Gaussian random fields on some metric measure space T and λ is a reference measure. It is easy to see that the measures μ_n form a martingale and converge in the weak*-sense to a limit measure μ that we call the *multiplicative chaos* associated to the sequence X_k. However, the 1st nontrivial question in the theory is whether μ is almost surely 0 or not.

In the Euclidean setting Kahane identified the log-correlated random fields to be the edge case when it comes to the non-triviality of the resulting chaos. We say that the Gaussian random field $X = \sum_{k=1}^{\infty} X_k$ is log-correlated if it formally has a covariance of the form

$$\mathbb{E}\, X(x)X(y) = \beta^2 \log \frac{1}{|x-y|} + g(x,y),$$

where g is a bounded and continuous function and $\beta > 0$ is a constant. The parameter β is often called the *inverse temperature* in the mathematical physics literature. The non-triviality of the chaos measure μ then depends on β and the dimension d of the space; if $0 < \beta < \sqrt{2d}$, μ is almost surely nontrivial; if $\beta \geq \sqrt{2d}$, it is almost surely 0. The regime $0 < \beta < \sqrt{2d}$ is called *subcritical*, while the regimes $\beta = \sqrt{2d}$ and $\beta > \sqrt{2d}$ are called *critical* and *supercritical*, respectively. In the critical and supercritical cases it is still possible to get nontrivial measures by performing a suitable renormalization [6, 7, 13].

The study of Gaussian multiplicative chaos in the log-correlated case has spurred a lot of interest, and comprehensive reviews exist [17, 18]. In this situation there are also results on uniqueness and convergence under different approximations [9, 17, 19, 22] as well as optimal results on L^p-boundedness [11].

In the non-Gaussian case the research has been less active. One model that has been studied is the product of random pulses [3, 4], which is a cascade-type construction where the b-adic grid has been replaced with intervals sampled using a Poisson process. The rest of the research has so far focused mainly on infinitely divisible processes. The paper [1] studies chaoses that are defined using a cone construction with an infinitely divisible independently scattered random measure on the upper half plane. A more recent paper [16] deals with chaoses that are $\star$-scale invariant, a specific class that again implies infinitely divisibility under some small assumptions. Finally, in [21] a field obtained from a statistical model of the Riemann ζ-function on the critical line is studied. The resulting chaos measure in this case is almost surely absolutely continuous with respect to a Gaussian chaos measure.

In this paper we will study the multiplicative chaos of non-Gaussian locally log-correlated random fields defined on the closed unit cube $I = [0,1]^d \subset \mathbf{R}^d$, see Definition 4 below. (Restricting to the unit cube is done purely on practical grounds and the results generalize easily to other domains.) Let X_k $(k = 1, 2, \ldots)$ be real-valued, continuous, independent, and centered random fields on I and $\beta > 0$ be a parameter. We define a sequence of measures on I by setting

$$\mu_n(f;\beta) = \int_I f(x) \frac{e^{\beta \sum_{k=1}^n X_k(x)}}{\mathbb{E}\, e^{\beta \sum_{k=1}^n X_k(x)}} \,\mathrm{d}x \tag{1}$$

for all $f \in C(I, \mathbf{R})$. Since we are not anymore in the Gaussian case, we must make the extra assumption that $\mathbb{E}\, e^{\beta X_k(x)}$ exists for all $k \geq 1$. Still, $\mu_n(f;\beta)$ is a martingale for all fixed $f \in C(I, \mathbf{R})$ and $\beta > 0$, and one gets the almost sure weak*-convergence as in Kahane's theory. Again, the crucial question is whether the limit is nontrivial or not.

We will in fact allow β to take complex values, in which case it becomes important to ensure that the denominator $\mathbb{E}\, e^{\beta \sum_{k=1}^n X_k(x)}$ does not vanish. Clearly, for any finite n it is possible to pick a neighborhood of $(0, \sqrt{2d})$ (which will be our region of interest) where this is true. The assumptions that will follow will ultimately ensure that one can choose such a neighborhood in such a way that this holds for all $n \geq 1$ simultaneously. It is worth noting that in the case of complex β we do not expect the limit μ to be a random measure but rather a distribution.

As an interesting example of a non-Gaussian locally log-correlated random field, consider the random Fourier series $\sum_{k=1}^\infty X_k(x)$, where

$$X_k(x) = \frac{1}{\sqrt{k}}(A_k \cos(2\pi k x) + B_k \sin(2\pi k x)), \quad x \in [0,1]. \tag{2}$$

Here A_k and B_k are i.i.d. centered random variables with variance 1. In this case we have the following theorem.

Theorem 1. Assume that the fields X_k are defined as in (2) and that $\mathbb{E}\, e^{\lambda A_1} < \infty$ for all $\lambda \in \mathbf{R}$. Then there exists an open set $U \subset \mathbf{C}$ containing the interval $(0, \sqrt{2d})$ such that for any compact $K \subset U$ there exists $p > 1$ for which the martingale $\mu_n(f;\beta)$ converges in $L^p(\Omega)$ for all $\beta \in K$ and $f \in C(I, \mathbf{C})$.

As a corollary, for a fixed $f \in C(I, \mathbf{C})$ the maps $\beta \mapsto \mu_n(f;\beta)$ converge almost surely uniformly on compact subsets of U to an analytic map $\beta \mapsto \mu(f;\beta)$.

Remark 2. For $\beta \in \mathbf{R}$ a standard argument using the fact that $C(I, \mathbf{R})$ is separable shows that the maps $\beta \mapsto \mu_n(f;\beta)$ converge almost surely for all $f \in C(I)$ simultaneously, and as a limit one obtains a random measure $\mu(\cdot;\beta)$, which is a continuous function of β in the weak*-topology of measures. By Kolmogorov's zero–one law for real β the total mass

$\mu(I;\beta)$ is almost surely nonzero when $|\beta| < \sqrt{2d}$. When $|\beta| > \sqrt{2d}$, a simple argument shows that the measure is almost surely 0, see Lemma 15 at the end of Section 2. Moreover, for a fixed $\beta \in U$ one can show that μ_n converges in $W^s(\mathbf{T}^d)$ for $s < -d$, where $W^s(\mathbf{T}^d)$ is the L^2-Sobolev space on the d-dimensional torus, which we identify with $[0,1)^d$, see Lemma 14.

Remark 3. Instead of defining the chaos distribution using the Lebesgue measure one could in principle use any Radon measure. Our methods apply also in this situation, but then the range of β for which one gets uniform integrability depends on the measure (and for some measures such as the Dirac delta the range of uniform integrability can be empty, assuming that we exclude the case $\beta = 0$).

The extension to the complex case is quite nontrivial in this situation, since we are missing local independence of the increments X_k, and hence the previously known methods for proving analyticity of the chaos fail completely. Here we develop a new method inspired by the clever and simple recent approach due to Berestycki [5]. The method of [5] does not, however, apply to our case directly. Indeed, in our proof we completely bypass the L^1-estimates, performing instead a more complicated but direct estimate in L^p via a dyadic analysis of the field. As a further distinction to [5], one may note that Girsanov's lemma is not applicable in the non-Gaussian setting.

Theorem 1 is a corollary of a more general result, which we state next.

Definition 4. We say that the sequence $(X_k)_{k=1}^{\infty}$ has a locally log-correlated structure if the following hold:

- We have $\sup_{x\in I} \mathbb{E}\, X_k(x)^2 \to 0$ and $\sum_{k=1}^{\infty} \mathbb{E}\, X_k(0)^2 = \infty$.
- There exists a constant $\delta > 0$ such that for all $n \geq 1$ and $x, y \in I$ satisfying $|x-y| \leq \delta$ we have

$$|\sum_{k=1}^{n} \mathbb{E}\, X_k(x)X_k(y) - \min\left(\log \frac{1}{|x-y|}, \sum_{k=1}^{n} \mathbb{E}\, X_k(0)^2\right)| \leq C \tag{3}$$

  for some constant $C > 0$.

In addition to having a locally log-correlated structure, we will require certain regularity of the fields X_k, which we list as conditions (4) and (5);

$$\sup_{x\in I} \sum_{k=1}^{\infty} \left(\mathbb{E}\, |X_k(x)|^{3+\varepsilon}\right)^{\frac{3}{3+\varepsilon}} < \infty \quad \text{for some } \varepsilon > 0 \tag{4}$$

$$\mathbb{E}\left|\sum_{k=1}^{n}(X_k(x)-X_k(y))\right|^r \le C_r e^{r\sum_{k=1}^{n}\mathbb{E}X_k(0)^2}|x-y|^r \quad \text{for } n\ge 1, \text{ and } r\ge 1. \tag{5}$$

In condition (5) the constant C_r is allowed to depend on r. We note that (5) can be deduced from either of the following two conditions:

$$\sum_{k=1}^{n}\mathbb{E}\,|X_k(x)-X_k(y)|^r \le C_r e^{r\sum_{k=1}^{n}\mathbb{E}X_k(0)^2}|x-y|^r \quad \text{for } n\ge 1, \text{and } r\ge 2 \tag{6}$$

$$\sum_{k=1}^{n}\mathbb{E}\sup_{x\in I}|X_k'(x)|^r \le C_r e^{r\sum_{k=1}^{n}\mathbb{E}X_k(0)^2} \quad \text{for } n\ge 1, \text{and } r\ge 2. \tag{7}$$

Indeed, the mean value theorem shows that (7) implies (6), while Rosenthal's inequality [20] shows that (6) implies (5). Note that in condition (7) we implicitly assume that $X_k(x)$ is almost surely continuously differentiable on I.

Theorem 5. Assume that $(X_k)_{k=1}^{\infty}$ is a sequence of independent, centered, and continuous random fields having a locally log-correlated structure and satisfying (4) and (5). Assume further that

$$\sup_{k\ge 1}\sup_{x\in I}\mathbb{E}\,e^{\lambda X_k(x)} < \infty \tag{8}$$

for all $\lambda \in \mathbf{R}$. Then the conclusions of Theorem 1 hold for the measures μ_n.

The proof of Theorem 5 is given in Section 2, and a brief outline is as follows: we start by normalizing the situation in such a way that the variance of the field on the level n is approximately $n\log(2)$. For each $x\in I$ we focus on the last level l on which the field is exceptionally large. For points sharing a common level l we perform a splitting of I into dyadic cubes with side length approximately 2^{-l} and derive an L^2-estimate in each of these cubes, conditioned on the level l. This takes care of the contribution coming from the tail of the field. After the conditional estimate we are still left with the contribution coming from level l, which is then handled by approximating the L^p-norm of the exponential of the supremum of the field, relying on the fact that the field being exceptionally large on level l is an event of low probability. Once we have established the boundedness in L^p, the rest of the proof is rather a routine.

In Section 3 we first prove Theorem 1, after which we provide another application of Theorem 5 where we consider a field that is the sum of dilated stationary processes, see Theorem 16. The latter example is related to the general model presented in [15].

2 Proof of Theorem 5

We start by splitting the field $\sum_{k=1}^{\infty} X_k(x)$ into blocks of approximately constant variances. For all $j \geq 0$ let $t_j \geq 1$ be the smallest index for which

$$\sum_{k=1}^{t_j} \mathbb{E}\, X_k(0)^2 \geq \log(2) j. \tag{9}$$

The t_j are well defined because of our assumption that $\sum_{k=1}^{\infty} \mathbb{E}\, X_k(0)^2 = \infty$, and we can use them to define the following auxiliary fields:

$$Y_j(x) = \sum_{k=t_{j-1}}^{t_j - 1} X_k(x), \quad Z_j(x) = \sum_{k=1}^{j} Y_k(x) \quad (j \geq 1).$$

Here we use the convention that $Y_j(x) = 0$ for all $x \in I$ if $t_{j-1} = t_j$, and we also set $Z_0(x) \equiv 0$.

The following lemma shows how the locally log-correlated structure of the fields X_k transfers to the fields Z_j.

Lemma 6. There exists a constant $C > 0$ such that for $|x - y| \leq \delta$ we have

$$\left| \mathbb{E}\, Z_j(x) Z_j(y) - \min\left(\log \frac{1}{|x-y|}, \log(2) j \right) \right| \leq C. \tag{10}$$

In particular the following inequalities hold for some constant $C > 0$:

$$|\mathbb{E}\, Z_j(x) Z_j(y) - \log(2) j| \leq C \quad (j \geq 1,\ |x-y| \leq 2^{-j}) \tag{11}$$

$$|\mathbb{E}\, Z_j(x) Z_j(y) - \mathbb{E}\, Z_m(x) Z_m(y)| \leq C \quad (j \geq m,\ 2^{-m-1} \leq |x-y| \leq \delta). \tag{12}$$

Proof. We have $\mathbb{E}\, Z_j(x) Z_j(y) = \sum_{k=1}^{j} \mathbb{E}\, Y_k(x) Y_k(y) = \sum_{k=1}^{t_j - 1} \mathbb{E}\, X_k(x) X_k(y)$. By (9) and the assumption that $\mathbb{E}\, X_k(0)^2 \to 0$, the variance on the level $t_j - 1$ satisfies

$$\sum_{k=1}^{t_j - 1} \mathbb{E}\, X_k(0)^2 = \log(2) j + O(1),$$

so (10) follows from (3). The inequalities (11) and (12) are easy corollaries. ∎

The next lemma provides a crucial estimate on the Laplace transform of the fields Z_j and it will be used extensively in the proofs. The idea behind it is the following: since $\mathbb{E}\, X_k(x)^2$ tends to 0, the constant variance increments Y_j will start to look like

Gaussians by the central limit theorem. This leads one to expect that some sort of Gaussianity appears also in the fields Z_j and here we quantify this for the Laplace transform of the vector $(Z_j(x), Z_j(y))$, where $x, y \in I$.

Lemma 7. Let $R > 0$. There exists $r = r(R) > 0$ such that if $\xi_1, \xi_2 \in \mathbf{C}$ satisfy $|\operatorname{Re} \xi_i| \le R$ and $|\operatorname{Im} \xi_i| \le r$ for $i = 1, 2$, then

$$\mathbb{E}\, e^{\xi_1 Z_j(x) + \xi_2 Z_j(y)} = e^{\frac{\xi_1^2}{2}\mathbb{E}\, Z_j(x)^2 + \frac{\xi_2^2}{2}\mathbb{E}\, Z_j(y)^2 + \xi_1\xi_2 \mathbb{E}\, Z_j(x)Z_j(y) + \varepsilon}, \tag{13}$$

where the error $\varepsilon = \varepsilon(\xi_1, \xi_2)$ is bounded and $|\varepsilon| \le C_R$ for some constant $C_R > 0$ depending on R.

Proof. Let $K = [-R, R] \times [-r, r]$, where $r \in (0, 1]$ will be chosen later. Define $\varphi_k(\xi_1, \xi_2) = \mathbb{E}\, e^{\xi_1 X_k(x) + \xi_2 X_k(y)}$. Then Taylor's theorem gives us

$$\varphi_k(\xi_1, \xi_2) = 1 + \frac{\mathbb{E}\, X_k(x)^2}{2}\xi_1^2 + \frac{\mathbb{E}\, X_k(y)^2}{2}\xi_2^2 + \mathbb{E}\, X_k(x)X_k(y)\xi_1\xi_2 + c_k O(|\xi|^3),$$

where

$$\begin{aligned} c_k = & \sup_{\xi_1,\xi_2 \in K} |\frac{\partial^3}{\partial \xi_1^3}\varphi_k(\xi_1, \xi_2)| + \sup_{\xi_1,\xi_2 \in K} |\frac{\partial^3}{\partial \xi_2^3}\varphi_k(\xi_1, \xi_2)| \\ & + \sup_{\xi_1,\xi_2 \in K} |\frac{\partial^3}{\partial \xi_1^2 \partial \xi_2}\varphi_k(\xi_1, \xi_2)| + \sup_{\xi_1,\xi_2 \in K} |\frac{\partial^3}{\partial \xi_1 \partial \xi_2^2}\varphi_k(\xi_1, \xi_2)|. \end{aligned}$$

We have for all $a, b \in \{0, 1, 2, 3\}$, $a + b = 3$ that

$$|\frac{\partial^3}{\partial \xi_1^a \xi_2^b}\varphi_k(\xi_1, \xi_2)| \le \mathbb{E}\, |X_k(x)|^a |X_k(y)|^b |e^{\xi_1 X_k(x) + \xi_2 X_k(y)}|,$$

which by Hölder's inequality is less than

$$\Big(\mathbb{E}\, |X_k(x)|^{3+\varepsilon}\Big)^{\frac{a}{3+\varepsilon}} \Big(\mathbb{E}\, |X_k(y)|^{3+\varepsilon}\Big)^{\frac{b}{3+\varepsilon}} \Big(\mathbb{E}\, |e^{\frac{3+\varepsilon}{\varepsilon/2}\xi_1 X_k(x)}|\Big)^{\frac{\varepsilon/2}{3+\varepsilon}} \Big(\mathbb{E}\, |e^{\frac{3+\varepsilon}{\varepsilon/2}\xi_2 X_k(x)}|\Big)^{\frac{\varepsilon/2}{3+\varepsilon}},$$

where ε is given by the assumption (4). The last two factors are bounded by the assumption (8). Finally, because

$$\Big(\mathbb{E}\, |X_k(x)|^{3+\varepsilon}\Big)^{\frac{a}{3+\varepsilon}} \Big(\mathbb{E}\, |X_k(y)|^{3+\varepsilon}\Big)^{\frac{b}{3+\varepsilon}} \le \frac{a\Big(\mathbb{E}\, |X_k(x)|^{3+\varepsilon}\Big)^{\frac{3}{3+\varepsilon}} + b\Big(\mathbb{E}\, |X_k(y)|^{3+\varepsilon}\Big)^{\frac{3}{3+\varepsilon}}}{3},$$

we see that

$$\sum_{k=1}^{\infty} c_k < \infty \tag{14}$$

by (4). Because $\sup_{x\in I}\mathbb{E}\,X_k(x)^2 \to 0$, there exists $k_0 \geq 1$ such that for large enough $k \geq k_0$ we have

$$\Big|\frac{\mathbb{E}\,X_k(x)^2}{2}\xi_1^2 + \frac{\mathbb{E}\,X_k(y)^2}{2}\xi_2^2 + \mathbb{E}\,X_k(x)X_k(y)\xi_1\xi_2 + c_k O(|\xi|^3)\Big| < \frac{1}{2}$$

whenever $\xi_1, \xi_2 \in K$. In particular if $\log\colon \mathbf{C}\setminus(-\infty,0] \to \mathbf{C}$ is the branch of the logarithm that takes the value 0 at 1, we have

$$\log(\varphi_k(\xi_1,\xi_2)) = \frac{\mathbb{E}\,X_k(x)^2}{2}\xi_1^2 + \frac{\mathbb{E}\,X_k(y)^2}{2}\xi_2^2 + \mathbb{E}\,X_k(x)X_k(y)\xi_1\xi_2 + d_k O(|\xi|^3 + |\xi|^6).$$

Here

$$\begin{aligned} d_k = {} & c_k + (\mathbb{E}\,X_k(x)^2)^2 + (\mathbb{E}\,X_k(y)^2)^2 + (\mathbb{E}\,X_k(x)X_k(y))^2 + c_k^2 \\ & + (\mathbb{E}\,X_k(x)^2 + \mathbb{E}\,X_k(y)^2 + |\mathbb{E}\,X_k(x)X_k(y)|)c_k + \mathbb{E}\,X_k(x)^2\mathbb{E}\,X_k(y)^2 \\ & + \mathbb{E}\,X_k(x)^2|\mathbb{E}\,X_k(x)X_k(y)| + \mathbb{E}\,X_k(y)^2|\mathbb{E}\,X_k(x)X_k(y)|. \end{aligned}$$

By Hölder's inequality and (4) we have

$$\sup_{x\in I}\sum_{k=1}^{\infty}(\mathbb{E}\,X_k(x)^2)^2 \leq \sup_{x\in I}\sum_{k=1}^{\infty}(\mathbb{E}\,|X_k(x)|^{3+\varepsilon})^{\frac{4}{3+\varepsilon}} < \infty,$$

and this together with (14) and Hölder's inequality gives $\sum_{k=1}^{\infty} d_k < \infty$. Now if $\varphi(\xi_1,\xi_2) = \mathbb{E}\,e^{\xi_1 Z_j(x)+\xi_2 Z_j(y)}$, we have by independence that

$$\varphi(\xi_1,\xi_2) = \prod_{k=1}^{k_0-1}\varphi_k(\xi_1,\xi_2)\prod_{k=k_0}^{t_j-1}\varphi_k(\xi_1,\xi_2).$$

By continuity r can be chosen so small that the absolute value of the 1st product is bounded from below and from above for all $\xi_1,\xi_2 \in K$, and hence the product can be swallowed into the constant C_R. The logarithm of the 2nd product is

$$\begin{aligned} & \sum_{k=k_0}^{t_j-1}\Big(\frac{\mathbb{E}\,X_k(x)^2}{2}\xi_1^2 + \frac{\mathbb{E}\,X_k(y)^2}{2}\xi_2^2 + \mathbb{E}\,X_k(x)X_k(y)\xi_1\xi_2 + d_k O(|\xi|^3 + |\xi|^6)\Big) \\ & = \frac{\xi_1^2}{2}\mathbb{E}\,Z_j(x)^2 + \frac{\xi_2^2}{2}\mathbb{E}\,Z_j(y)^2 + \xi_1\xi_2\mathbb{E}\,Z_j(x)Z_j(y) - \frac{\xi_1^2}{2}\sum_{k=1}^{k_0-1}\mathbb{E}\,X_k(x)^2 \\ & - \frac{\xi_2^2}{2}\sum_{k=1}^{k_0-1}\mathbb{E}\,X_k(y)^2 - \xi_1\xi_2\sum_{k=1}^{k_0-1}\mathbb{E}\,X_k(x)X_k(y) + O(|\xi|^3 + |\xi|^6) \end{aligned}$$

and everything but the 1st three terms can be put in C_R. ∎

To prepare for the proof of Theorem 5 we start by fixing some notation and then prove the main estimates as lemmas. First of all, we assume that $\delta > \sqrt{d}$, so that the estimates (10) and (12) hold for all $x, y \in I$. We will show how to get rid of this assumption in the end. Second, for a given β we let $\alpha \in (\operatorname{Re}\beta, 2\operatorname{Re}\beta)$ be a fixed real parameter. We will not specify the exact value of α, but it will be clear from the proof that choosing it sufficiently close to $\operatorname{Re}\beta$ will work. We assume that $\operatorname{Re}\beta \in (0, \sqrt{2d})$ and that

$$(\operatorname{Im}\beta)^2 < \min\left\{\frac{(\alpha - \operatorname{Re}\beta)^2}{2}, \frac{(2\operatorname{Re}\beta - \alpha)^2}{2} - (\operatorname{Re}\beta)^2 + d, r^2\right\}, \tag{15}$$

where r is obtained from Lemma 7 applied with $R = 2\sqrt{2d}$. Notice that by choosing α close enough to $\operatorname{Re}\beta$ it is always possible to have $\frac{(2\operatorname{Re}\beta-\alpha)^2}{2} - (\operatorname{Re}\beta)^2 + d > 0$.

For $l \geq 0$ we let

$$A_l(x) = \{Z_l(x) \geq \alpha \mathbb{E}\, Z_l(x)^2\}$$

be the event that Z_l is exceptionally large at the point $x \in I$. Similarly, for $k \geq l$ we let

$$B_{l,k}(x) = \{Z_j(x) < \alpha \mathbb{E}\, Z_j(x)^2 \text{ for all } l+1 \leq j \leq k\}$$

be the event that Z_j is small from level $l+1$ to level k.

To keep formulas short (or at least shorter), define

$$E_k(x) = \frac{e^{\beta Z_k(x)}}{\mathbb{E}\, e^{\beta Z_k(x)}}$$

for all $k \geq 1$, together with the notation

$$\begin{aligned} Z_{(m,k]}(x) &= Z_k(x) - Z_m(x) \\ E_{(m,k]}(x) &= \frac{e^{\beta Z_{(m,k]}(x)}}{\mathbb{E}\, e^{\beta Z_{(m,k]}(x)}} \\ A_{(m,k]}(x) &= \{Z_{(m,k]}(x) \geq \alpha \mathbb{E}\, Z_{(m,k]}(x)^2\} \end{aligned}$$

for all $0 \leq m \leq k$. Moreover, define

$$\mathcal{F}_m = \sigma(X_1, \ldots, X_{t_m - 1}).$$

Note that the variables with the subscript $(m,k]$ are independent of $\mathcal{F}_m$.

Finally, we use the notation $A \lesssim B$ to indicate that there exists a constant $C > 0$ only depending on α, β, d, and the distribution of the fields X_k such that the inequality $A \leq CB$ holds. We write $A \approx B$ when both $A \lesssim B$ and $B \lesssim A$ hold.

We start with a couple of technical lemmas.

Lemma 8. Let $k \geq m$. Then

$$\begin{aligned}&\mathbb{E}\,[|E_{(m,k]}(x)\overline{E_{(m,k]}(y)}|\mathbf{1}_{A_{(m,k]}(x)}]\\&\lesssim e^{-\frac{(\alpha-\operatorname{Re}\beta)^2}{2}\log(2)(k-m)}e^{(\operatorname{Im}\beta)^2\log(2)(k-m)+\alpha(\operatorname{Re}\beta)\mathbb{E}\,[Z_{(m,k]}(x)Z_{(m,k]}(y)]}.\end{aligned}$$

Proof. By Lemma 7 we have

$$\begin{aligned}|E_{(m,k]}(x)\overline{E_{(m,k]}(y)}| &= \frac{e^{(\operatorname{Re}\beta)(Z_{(m,k]}(x)+Z_{(m,k]}(y))}}{|\mathbb{E}\,e^{\beta Z_{(m,k]}(x)}||\mathbb{E}\,e^{\overline{\beta}Z_{(m,k]}(y)}|}\\&\lesssim e^{\frac{(\operatorname{Im}\beta)^2-(\operatorname{Re}\beta)^2}{2}(\mathbb{E}\,Z_{(m,k]}(x)^2+\mathbb{E}\,Z_{(m,k]}(y)^2)}e^{(\operatorname{Re}\beta)(Z_{(m,k]}(x)+Z_{(m,k]}(y))}.\end{aligned}\tag{16}$$

Moreover,

$$\begin{aligned}&\mathbb{E}\,[e^{(\operatorname{Re}\beta)(Z_{(m,k]}(x)+Z_{(m,k]}(y))}\mathbf{1}_{A_{(m,k]}(x)}]\\&\leq \mathbb{E}\,[e^{(\operatorname{Re}\beta)(Z_{(m,k]}(x)+Z_{(m,k]}(y))}e^{(\alpha-\operatorname{Re}\beta)Z_{(m,k]}(x)-(\alpha-\operatorname{Re}\beta)\alpha\mathbb{E}\,Z_{(m,k]}(x)^2}]\\&= \mathbb{E}\,[e^{\alpha Z_{(m,k]}(x)+(\operatorname{Re}\beta)Z_{(m,k]}(y)-\alpha^2\mathbb{E}\,Z_{(m,k]}(x)^2+\alpha(\operatorname{Re}\beta)\mathbb{E}\,Z_{(m,k]}(x)^2}]\\&\lesssim e^{-\frac{\alpha^2}{2}\mathbb{E}\,Z_{(m,k]}(x)^2+\alpha(\operatorname{Re}\beta)\mathbb{E}\,Z_{(m,k]}(x)^2+\frac{(\operatorname{Re}\beta)^2}{2}\mathbb{E}\,Z_{(m,k]}(y)^2+\alpha(\operatorname{Re}\beta)\mathbb{E}\,[Z_{(m,k]}(x)Z_{(m,k]}(y)]},\end{aligned}$$

which together with the factor $e^{\frac{(\operatorname{Im}\beta)^2-(\operatorname{Re}\beta)^2}{2}(\mathbb{E}\,Z_{(m,k]}(x)^2+\mathbb{E}\,Z_{(m,k]}(y)^2)}$ gives us

$$\begin{aligned}&\mathbb{E}\,[|E_{(m,k]}(x)\overline{E_{(m,k]}(y)}|\mathbf{1}_{A_{(m,k]}(x)}]\\&\lesssim e^{-\frac{(\alpha-\operatorname{Re}\beta)^2}{2}\mathbb{E}\,Z_{(m,k]}(x)^2}e^{\frac{(\operatorname{Im}\beta)^2}{2}(\mathbb{E}\,Z_{(m,k]}(x)^2+\mathbb{E}\,Z_{(m,k]}(y)^2)+\alpha(\operatorname{Re}\beta)\mathbb{E}\,[Z_{(m,k]}(x)Z_{(m,k]}(y)]},\end{aligned}$$

from which the claim follows by Lemma 6. ■

The following lemma is used in the proof of Proposition 11 below to handle the tail of the field for points $x, y \in I$ that are far enough from each other.

Lemma 9. Assume that $|x-y| \geq 2^{-m-1}$. Then for all $n \geq m$ we have

$$\mathbf{1}_{B_{m-1,m}(x)}\mathbf{1}_{B_{m-1,m}(y)}|\mathbb{E}\,[E_{(m,n]}(x)\overline{E_{(m,n]}(y)}\mathbf{1}_{B_{m,n}(x)}\mathbf{1}_{B_{m,n}(y)}|\mathcal{F}_m]| \lesssim 1.$$

Proof. Define

$$P_k = E_{(m,k]}(x)\overline{E_{(m,k]}(y)}\mathbf{1}_{B_{m,k}(x)}\mathbf{1}_{B_{m,k}(y)}$$

for $k \geq m$. Then

$$\begin{aligned}
P_{k+1} &= E_{(m,k+1]}(x)\overline{E_{(m,k+1]}(y)}\mathbf{1}_{B_{m,k}(x)}\mathbf{1}_{B_{m,k}(y)}\mathbf{1}_{B_{k,k+1}(x)}\mathbf{1}_{B_{k,k+1}(y)} \\
&= -E_{(m,k+1]}(x)\overline{E_{(m,k+1]}(y)}\mathbf{1}_{B_{m,k}(x)}\mathbf{1}_{B_{m,k}(y)}(1-\mathbf{1}_{B_{k,k+1}(x)}\mathbf{1}_{B_{k,k+1}(y)}) \\
&\quad + \frac{e^{\beta Y_{k+1}(x)}}{\mathbb{E}\, e^{\beta Y_{k+1}(x)}} \cdot \frac{e^{\overline{\beta} Y_{k+1}(y)}}{\mathbb{E}\, e^{\overline{\beta} Y_{k+1}(y)}} P_k.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\mathbf{1}_{\{Z_m(x)<\alpha \mathbb{E}\, Z_m(x)^2\}}\mathbf{1}_{\{Z_m(y)<\alpha \mathbb{E}\, Z_m(y)^2\}}|\mathbb{E}\,[P_{k+1}|\mathcal{F}_m]| \\
&\quad \leq \mathbf{1}_{\{Z_m(x)<\alpha \mathbb{E}\, Z_m(x)^2\}}\mathbb{E}\,[|E_{(m,k+1]}(x)\overline{E_{(m,k+1]}(y)}|\mathbf{1}_{A_{k+1}(x)}|\mathcal{F}_m] \\
&\quad + \mathbf{1}_{\{Z_m(y)<\alpha \mathbb{E}\, Z_m(y)^2\}}\mathbb{E}\,[|E_{(m,k+1]}(x)\overline{E_{(m,k+1]}(y)}|\mathbf{1}_{A_{k+1}(y)}|\mathcal{F}_m] \\
&\quad + \frac{|\mathbb{E}\, e^{\beta Y_{k+1}(x)+\overline{\beta} Y_{k+1}(y)}|}{|\mathbb{E}\, e^{\beta Y_{k+1}(x)}\mathbb{E}\, e^{\overline{\beta} Y_{k+1}(y)}|}\mathbf{1}_{\{Z_m(x)<\alpha \mathbb{E}\, Z_m(x)^2\}}\mathbf{1}_{\{Z_m(y)<\alpha \mathbb{E}\, Z_m(y)^2\}}|\mathbb{E}\,[P_k|\mathcal{F}_m]|.
\end{aligned} \tag{17}$$

Notice that $A_{k+1}(x) \cap \{Z_m(x) < \alpha \mathbb{E}\, Z_m(x)^2\} \subset A_{(m,k+1]}(x)$. This and Lemmas 8 and 6 give us

$$\begin{aligned}
&\mathbf{1}_{\{Z_m(x)<\alpha \mathbb{E}\, Z_m(x)^2\}}\mathbb{E}\,[|E_{(m,k+1]}(x)\overline{E_{(m,k+1]}(y)}|\mathbf{1}_{A_{k+1}(x)}|\mathcal{F}_m] \\
&\leq \mathbb{E}\,[|E_{(m,k+1]}(x)\overline{E_{(m,k+1]}(y)}|\mathbf{1}_{A_{(m,k+1]}(x)}] \\
&\lesssim e^{-\frac{(\alpha-\operatorname{Re}\beta)^2}{2}\log(2)(k+1-m)} e^{(\operatorname{Im}\beta)^2\log(2)(k+1-m)+\alpha(\operatorname{Re}\beta)\mathbb{E}\,[Z_{(m,k+1]}(x)Z_{(m,k+1]}(y)]} \\
&\lesssim e^{(-\frac{(\alpha-\operatorname{Re}\beta)^2}{2}+(\operatorname{Im}\beta)^2)\log(2)(k+1-m)} =: \sigma_{k+1}.
\end{aligned}$$

Similarly, we get

$$\mathbf{1}_{\{Z_m(y)<\alpha \mathbb{E}\, Z_m(y)^2\}}\mathbb{E}\,[|E_{(m,k+1]}(x)\overline{E_{(m,k+1]}(y)}|\mathbf{1}_{A_{k+1}(y)}|\mathcal{F}_m] \lesssim \sigma_{k+1}$$

and because of (15) the σ_k decay exponentially. Moving on to the 3rd term on the right-hand side of (17), let

$$\frac{|\mathbb{E}\, e^{\beta Y_{k+1}(x)+\overline{\beta} Y_{k+1}(y)}|}{|\mathbb{E}\, e^{\beta Y_{k+1}(x)}\mathbb{E}\, e^{\overline{\beta} Y_{k+1}(y)}|} =: \rho_{k+1}.$$

By Lemma 6 there exists a constant A such that for all $m \leq m' \leq M$ we have

$$\begin{aligned}
\prod_{k=m'+1}^{M}\rho_k &= \frac{|\mathbb{E}\, e^{\beta Z_{(m',M]}(x)+\overline{\beta} Z_{(m',M]}(y)}|}{|\mathbb{E}\, e^{\beta Z_{(m',M]}(x)}||\mathbb{E}\, e^{\overline{\beta} Z_{(m',M]}(y)}|} \\
&= \frac{|\mathbb{E}\, e^{\beta Z_M(x)+\overline{\beta} Z_M(y)}||\mathbb{E}\, e^{\beta Z_{m'}(x)}||\mathbb{E}\, e^{\overline{\beta} Z_{m'}(y)}|}{|\mathbb{E}\, e^{\beta Z_M(x)}||\mathbb{E}\, e^{\overline{\beta} Z_M(y)}||\mathbb{E}\, e^{\beta Z_{m'}(x)+\overline{\beta} Z_{m'}(y)}|} \\
&\leq c e^{|\beta|^2(\mathbb{E}\, Z_M(x)Z_M(y)-\mathbb{E}\, Z_{m'}(x)Z_{m'}(y))} \leq A,
\end{aligned}$$

where $c > 0$ is a constant arising from Lemma 7. We have thus verified that

$$\begin{aligned}&\mathbf{1}_{B_{m-1,m}(x)}\mathbf{1}_{B_{m-1,m}(y)}|\mathbb{E}\,[P_{k+1}|\mathcal{F}_m]|\\ &\quad\le C\sigma_{k+1}+\rho_{k+1}\mathbf{1}_{B_{m-1,m}(x)}\mathbf{1}_{B_{m-1,m}(y)}|\mathbb{E}\,[P_k|\mathcal{F}_m]|\end{aligned}$$

for some constant $C > 0$, and it is easy to see that we then have

$$\mathbf{1}_{\{Z_m(x)<\alpha\mathbb{E}\,Z_m(x)^2\}}\mathbf{1}_{\{Z_m(y)<\alpha\mathbb{E}\,Z_m(y)^2\}}|\mathbb{E}\,[P_k|\mathcal{F}_m] \le CA(1+\sum_{j=m+1}^{\infty}\sigma_j)$$

for all $k \ge m$. Taking $k = n$ proves the claim. ∎

In the next lemma we bound the main contribution coming from points $x, y \in I$ that are at a given dyadic distance less than 2^{-m} from each other.

Lemma 10. Let $m \ge l+1$. If $|x-y| \le 2^{-m}$, we have

$$\begin{aligned}&\mathbf{1}_{A_l(x)}\mathbf{1}_{A_l(y)}\mathbb{E}\,[|E_{(l,m]}(x)\overline{E_{(l,m]}(y)}|\mathbf{1}_{B_{l,m}(x)}\mathbf{1}_{B_{l,m}(y)}|\mathcal{F}_l]\\ &\quad\lesssim e^{\left((\operatorname{Re}\beta)^2-\frac{(2\operatorname{Re}\beta-\alpha)^2}{2}\right)\log(2)(m-l)+(\operatorname{Im}\beta)^2\log(2)(m-l)}.\end{aligned}$$

Proof. By (16) and (11) we have

$$|E_{(l,m]}(x)\overline{E_{(l,m]}(y)}| \lesssim e^{(\operatorname{Re}\beta)(Z_{(l,m]}(x)+Z_{(l,m]}(y))+((\operatorname{Im}\beta)^2-(\operatorname{Re}\beta)^2)\log(2)(m-l)}.$$

On the other hand,

$$\begin{aligned}\mathbf{1}_{A_l(x)}\mathbf{1}_{B_{l,m}(x)} &\le \mathbf{1}_{\{Z_l(x)\ge\alpha\mathbb{E}\,Z_l(x)^2\}}\mathbf{1}_{\{Z_m(x)<\alpha\mathbb{E}\,Z_m(x)^2\}}\\ &\le \mathbf{1}_{\{Z_{(l,m]}(x)<\alpha\mathbb{E}\,Z_{(l,m]}(x)^2\}}\\ &\le e^{(\operatorname{Re}\beta-\frac{\alpha}{2})\alpha\mathbb{E}\,Z_{(l,m]}(x)^2-(\operatorname{Re}\beta-\frac{\alpha}{2})Z_{(l,m]}(x)}\end{aligned}$$

and similarly for $\mathbf{1}_{A_l(y)}\mathbf{1}_{B_{l,m}(y)}$. By Lemma 6 we therefore have

$$\begin{aligned}&\mathbf{1}_{A_l(x)}\mathbf{1}_{A_l(y)}\mathbb{E}\,[|E_{(l,m]}(x)\overline{E_{(l,m]}(y)}|\mathbf{1}_{B_{l,m}(x)}\mathbf{1}_{B_{l,m}(y)}|\mathcal{F}_l]\\ &\quad\lesssim e^{((\operatorname{Im}\beta)^2-(\operatorname{Re}\beta)^2)\log(2)(m-l)+(2\operatorname{Re}\beta-\alpha)\alpha\log(2)(m-l)}\mathbb{E}\,[e^{\frac{\alpha}{2}(Z_{(l,m]}(x)+Z_{(l,m]}(y))}]\\ &\quad\lesssim e^{((\operatorname{Im}\beta)^2-(\operatorname{Re}\beta)^2)\log(2)(m-l)+(2\operatorname{Re}\beta-\alpha)\alpha\log(2)(m-l)+\frac{\alpha^2}{2}\log(2)(m-l)},\end{aligned}$$

from which the claim follows. ∎

The following proposition encodes our fundamental L^2-estimate.

Proposition 11. Let J be a dyadic subcube of I with diameter at most 2^{-l-1}. Then for all $n \geq l$ and $f \in C(I, \mathbf{C})$ we have

$$\mathbb{E}\left[\left|\int_J f(x)E_n(x)\mathbf{1}_{A_l(x)}\mathbf{1}_{B_{l,n}(x)}\,\mathrm{d}x\right|^2 \middle| \mathcal{F}_l\right] \lesssim 2^{-2dl}\|f\|_\infty^2\big(\sup_{x\in J}|E_l(x)|^2\mathbf{1}_{A_l(x)}\big).$$

The constant in the inequality does not depend on J.

Proof. Let us partition the set J^2 into sets $\mathcal{D}_m$, $l+1 \leq m \leq n$ by setting

$$\mathcal{D}_m = \begin{cases} \{(x,y) \in J^2 : 2^{-m-1} \leq |x-y| \leq 2^{-m}\}, & \text{when } l+1 \leq m \leq n-1 \\ \{(x,y) \in J^2 : |x-y| \leq 2^{-n}\}, & \text{when } m = n. \end{cases}$$

Then one can write

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_J f(x)E_n(x)\mathbf{1}_{A_l(x)}\mathbf{1}_{B_{l,n}(x)}\,\mathrm{d}x\right|^2 \middle| \mathcal{F}_l\right] \\
&= \int_J\int_J f(x)\overline{f(y)}\,\mathbb{E}\left[E_n(x)\overline{E_n(y)}\mathbf{1}_{A_l(x)}\mathbf{1}_{A_l(y)}\mathbf{1}_{B_{l,n}(x)}\mathbf{1}_{B_{l,n}(y)}\middle|\mathcal{F}_l\right]\mathrm{d}x\,\mathrm{d}y \\
&= \int_J\int_J f(x)\overline{f(y)}E_l(x)\overline{E_l(y)}\mathbf{1}_{A_l(x)}\mathbf{1}_{A_l(y)} \\
&\quad\times \mathbb{E}\left[E_{(l,n]}(x)\overline{E_{(l,n]}(y)}\mathbf{1}_{B_{l,n}(x)}\mathbf{1}_{B_{l,n}(y)}\middle|\mathcal{F}_l\right]\mathrm{d}x\,\mathrm{d}y \\
&= \sum_{m=l+1}^{n}\int_{(x,y)\in\mathcal{D}_m} f(x)\overline{f(y)}E_l(x)\overline{E_l(y)}\mathbf{1}_{A_l(x)}\mathbf{1}_{A_l(y)} \\
&\quad\times \mathbb{E}\Big[E_{(l,m]}(x)\overline{E_{(l,m]}(y)}\mathbf{1}_{B_{l,m}(x)}\mathbf{1}_{B_{l,m}(y)} \\
&\quad\times \mathbb{E}\left[E_{(m,n]}(x)\overline{E_{(m,n]}(y)}\mathbf{1}_{B_{m,n}(x)}\mathbf{1}_{B_{m,n}(y)}\middle|\mathcal{F}_m\right]\Big|\mathcal{F}_l\Big]\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \|f\|_\infty^2\big(\sup_{x\in J}|E_l(x)|^2\mathbf{1}_{A_l(x)}\big) \\
&\quad\times \sum_{m=l+1}^{n}\int_{(x,y)\in\mathcal{D}_m}\mathbf{1}_{A_l(x)}\mathbf{1}_{A_l(y)}\mathbb{E}\Big[\left|E_{(l,m]}(x)\overline{E_{(l,m]}(y)}\right|\mathbf{1}_{B_{l,m}(x)}\mathbf{1}_{B_{l,m}(y)} \\
&\quad\times \left|\mathbb{E}\left[E_{(m,n]}(x)\overline{E_{(m,n]}(y)}\mathbf{1}_{B_{m,n}(x)}\mathbf{1}_{B_{m,n}(y)}\middle|\mathcal{F}_m\right]\right|\Big|\mathcal{F}_l\Big]\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Here we use the convention that $E_{(n,n]}(x) = E_{(n,n]}(y) = \mathbf{1}_{B_{n,n}(x)} = \mathbf{1}_{B_{n,n}(y)} = 1$. By Lemmas 9 and 10, and (15) we have

$$\begin{aligned}
&\sum_{m=l+1}^{n} \int_{(x,y)\in\mathcal{D}_m} \mathbf{1}_{A_l(x)}\mathbf{1}_{A_l(y)}\mathbb{E}\left[|E_{(l,m]}(x)\overline{E_{(l,m]}(y)}|\mathbf{1}_{B_{l,m}(x)}\mathbf{1}_{B_{l,m}(y)}\times\right.\\
&\quad\left.|\mathbb{E}\,[E_{(m,n]}(x)\overline{E_{(m,n]}(y)}\mathbf{1}_{B_{m,n}(x)}\mathbf{1}_{B_{m,n}(y)}|\mathcal{F}_m]|\big|\mathcal{F}_l\right]\\
&\lesssim \sum_{m=l+1}^{n} \int_{(x,y)\in\mathcal{D}_m} e^{\left((\operatorname{Re}\beta)^2-\frac{(2\operatorname{Re}\beta-\alpha)^2}{2}\right)\log(2)(m-l)+(\operatorname{Im}\beta)^2\log(2)(m-l)}\\
&\lesssim \sum_{m=l+1}^{n} 2^{-md-ld}e^{\left((\operatorname{Re}\beta)^2-\frac{(2\operatorname{Re}\beta-\alpha)^2}{2}\right)\log(2)(m-l)+(\operatorname{Im}\beta)^2\log(2)(m-l)}\\
&= 2^{-2ld}\sum_{j=1}^{n-l} e^{\left((\operatorname{Re}\beta)^2-\frac{(2\operatorname{Re}\beta-\alpha)^2}{2}\right)\log(2)j+(\operatorname{Im}\beta)^2\log(2)j-\log(2)jd}\\
&\lesssim 2^{-2ld}.
\end{aligned}$$

■

Next we wish to bound the term $\sup_{x\in J}|E_l(x)|\mathbf{1}_{A_l(x)}$ that appears after Proposition 11 has been applied. We come to our 2nd main estimate.

Proposition 12. For $p > 1$ sufficiently close to 1 we have

$$\mathbb{E}\sup_{x\in J}|E_l(x)|^p\mathbf{1}_{A_l(x)} \lesssim e^{-\varepsilon l}$$

for some $\varepsilon > 0$. Here J is a dyadic subcube of I with side length proportional to 2^{-l}. The estimate does not depend on J.

In the proof we need an estimate on $\mathbb{E}\,e^{\alpha\sup_{x\in J}Z_l(x)}$. We will write this estimate as a separate lemma, since it will also be used to show that the subcritical regime does not extend past $\sqrt{2d}$.

Lemma 13. Let $\gamma > 0$ be a real number and J a dyadic subcube of I with side length 2^{-l}. Then for any $u > 1$ there exists a constant $c > 0$ depending on u and γ but not on J or l, such that

$$\mathbb{E}\,e^{\gamma\sup_{x\in J}Z_l(x)} \le ce^{u\frac{\gamma^2}{2}\log(2)l}.$$

Proof. We will prove this using induction. Our induction hypothesis is that for all $u > 1$ there exists a constant $c > 0$ (which may depend on u and γ) such that

$$\sup_{x \in J_2} \mathbb{E}\, e^{\gamma \sup_{y \in J_1} Z_l(y,x)} \le c e^{u \frac{\gamma^2}{2} \log(2) l}, \tag{18}$$

where in the nth step of the induction $J = J_1 \times J_2$ with $J_1 \subset \mathbf{R}^n$ and $J_2 \subset \mathbf{R}^{d-n}$. The case $n = d$ is our final estimate.

When $n = 0$, (18) follows from Lemma 7. Assume then that $1 \le n \le d$ and that the claim holds for $n-1$. Write $J = J_1 \times L \times J_2$ with $J_1 \subset \mathbf{R}^{n-1}$, $L \subset \mathbf{R}$, and $J_2 \subset \mathbf{R}^{d-n}$. Let $x \in J_2$ be fixed and define the auxiliary process $\widetilde{Z}_l(t) = \sup_{y \in J_1} Z_l(y,t,x)$. For all $m \ge 0$ let D_m be the collection of dyadic subintervals of L with side length 2^{-l-m}. Choose a point $t_{m,i}$, $i = 1, \ldots, 2^m$, from each interval in D_m, and write $\pi(t_{m,i})$ for the point chosen from the parent interval of $t_{m,i}$. For example, one can take $t_{m,i}$ to be the center of its corresponding interval. Then for any fixed $t \in L$ there exists a sequence of points t_{m,i_m} converging to t such that t_{m,i_m} is chosen from inside the interval containing $t_{m-1,i_{m-1}}$. Now by the continuity of $\widetilde{Z}_l$ we have

$$\begin{aligned} e^{\gamma \widetilde{Z}_l(t)} &\le e^{\gamma \widetilde{Z}_l(t_{0,1})} + \sum_{m=1}^{\infty} |e^{\gamma \widetilde{Z}_l(t_{m,i_m})} - e^{\gamma \widetilde{Z}_l(t_{m-1,i_{m-1}})}| \\ &\le e^{\gamma \widetilde{Z}_l(t_{0,1})} + \sum_{m=1}^{\infty} \sup_{i \in \{1,\ldots,2^m\}} |e^{\gamma \widetilde{Z}_l(t_{m,i})} - e^{\gamma \widetilde{Z}_l(\pi(t_{m,i}))}|. \end{aligned}$$

Since the right-hand side does not depend on t, we have

$$\begin{aligned} e^{\gamma \sup_{t \in L} \widetilde{Z}_l(t)} &\le e^{\gamma \widetilde{Z}_l(t_{0,1})} + \sum_{m=1}^{\infty} \sup_{i \in \{1,\ldots,2^m\}} |e^{\gamma \widetilde{Z}_l(t_{m,i})} - e^{\gamma \widetilde{Z}_l(\pi(t_{m,i}))}| \\ &\le e^{\gamma \widetilde{Z}_l(t_{0,1})} + \sum_{m=1}^{\infty} \Big(\sum_{i=1}^{2^m} |e^{\gamma \widetilde{Z}_l(t_{m,i})} - e^{\gamma \widetilde{Z}_l(\pi(t_{m,i}))}|^r \Big)^{1/r} \\ &\le e^{\gamma \widetilde{Z}_l(t_{0,1})} + \sum_{m=1}^{\infty} \Big(\sum_{i=1}^{2^m} |\gamma \widetilde{Z}_l(t_{m,i}) - \gamma \widetilde{Z}_l(\pi(t_{m,i}))|^r \times \\ &\qquad (e^{r\gamma \widetilde{Z}_l(t_{m,i})} + e^{r\gamma \widetilde{Z}_l(\pi(t_{m,i}))}) \Big)^{1/r}, \end{aligned}$$

where $r > 1$ and we have used the elementary inequality

$$|e^a - e^b|^r \le |a-b|^r \sup_{a \le x \le b} e^{xr} \le |a-b|^r (e^{ar} + e^{br})$$

for any real numbers $a \le b$. Taking the expectation and using Jensen's inequality give

$$\mathbb{E}\, e^{\gamma \sup_{t\in L} \widetilde{Z}_l(t)} \le \mathbb{E}\, e^{\gamma \widetilde{Z}_l(t_{0,1})} + \gamma \sum_{m=1}^{\infty} \Big(\sum_{i=1}^{2^m} \mathbb{E}\big[|\widetilde{Z}_l(t_{m,i}) - \widetilde{Z}_l(\pi(t_{m,i}))|^r \times$$
$$(e^{r\gamma \widetilde{Z}_l(t_{m,i})} + e^{r\gamma \widetilde{Z}_l(\pi(t_{m,i}))}) \big] \Big)^{1/r}.$$

Moreover, by Hölder's inequality and the induction hypothesis we get for any $s > 1$ that

$$\begin{aligned}
&\mathbb{E}\big[|\widetilde{Z}_l(t_{m,i}) - \widetilde{Z}_l(\pi(t_{m,i}))|^r e^{r\gamma \widetilde{Z}_l(t_{m,i})} \big] \\
&\le \Big(\mathbb{E}\, |\widetilde{Z}_l(t_{m,i}) - \widetilde{Z}_l(\pi(t_{m,i}))|^{\frac{rs}{s-1}} \Big)^{\frac{s-1}{s}} \Big(\mathbb{E}\, e^{rs\gamma \widetilde{Z}_l(t_{m,i})} \Big)^{1/s} \\
&\le c \Big(\mathbb{E}\, |\widetilde{Z}_l(t_{m,i}) - \widetilde{Z}_l(\pi(t_{m,i}))|^{\frac{rs}{s-1}} \Big)^{\frac{s-1}{s}} e^{u' \frac{r^2 s \gamma^2}{2} \log(2) l}
\end{aligned}$$

for some $c > 0$. We will next use the inequality

$$\mathbb{E}\, |\widetilde{Z}_l(t) - \widetilde{Z}_l(t')|^p \le C_{p,\varepsilon} 2^{lp} |t - t'|^{p-\varepsilon}, \tag{19}$$

where $p > 1$ can be arbitrarily large and $\varepsilon > 0$ arbitrarily small, at the cost of the constant $C_{p,\varepsilon}$. The inequality is uniform in $x \in J_2$. We postpone proving (19) to the end of the proof. This gives us for any $\varepsilon > 0$ that

$$\mathbb{E}\big[|\widetilde{Z}_l(t_{m,i}) - \widetilde{Z}_l(\pi(t_{m,i}))|^r e^{r\gamma \widetilde{Z}_l(t_{m,i})} \big] \le C 2^{-mr + m\varepsilon + l\varepsilon} e^{u' \frac{r^2 s \gamma^2}{2} \log(2) l}$$

for some constant $C > 0$. A similar bound holds also for

$$\mathbb{E}\big[|\widetilde{Z}_l(t_{m,i}) - \widetilde{Z}_l(\pi(t_{m,i}))|^r e^{r\gamma \widetilde{Z}_l(\pi(t_{m,i}))} \big].$$

Thus, we have

$$\mathbb{E}\, e^{\gamma \sup_{t\in L} \widetilde{Z}_l(t)} \le C e^{u' \frac{\gamma^2}{2} \log(2) l} + C e^{u' \frac{rs\gamma^2}{2} \log(2) l} \sum_{m=1}^{\infty} 2^{\frac{1}{r}(m(1-r) + (m+l)\varepsilon)},$$

and choosing $r, s, u' > 1$ close enough to 1 and then ε small enough so that the series converges and $u' \frac{rs\gamma^2}{2} + \varepsilon \le u$ proves the claim (note that all the estimates used are uniform in $x \in J_2$).

It remains to show (19). We will again use a chaining argument, but this time in a higher-dimensional cube. Define yet another process $W(y) = Z_l(y, t, x) - Z_l(y, t', x)$ with

t, t', and x regarded as fixed. This time we divide the $n-1$-dimensional cube J_1 into dyadic subcubes. Let again D_m denote the cubes on the mth level, that is, the ones with side length 2^{-l-m}, and pick a point $y_{m,i} \in D_m$, $m \geq 0$, $1 \leq i \leq 2^{m(n-1)}$ from each cube with $\pi(y_{m,i})$ denoting the point of the parent cube of $y_{m,i}$. As before, we have

$$\begin{aligned}\sup_{y\in J_1} |W(y)| &\leq |W(y_{0,1})| + \sum_{m=1}^{\infty} \sup_{i\in\{1,\ldots,2^{m(n-1)}\}} |W(y_{m,i}) - W(\pi(y_{m,i}))| \\ &\leq |W(y_{0,1})| + \sum_{m=1}^{\infty} \Big(\sum_{i=1}^{2^{m(n-1)}} |W(y_{m,i}) - W(\pi(y_{m,i}))|^s \Big)^{1/s},\end{aligned}$$

where $s > 1$ is some number to be determined later. Let p be as in (19) and assume that $s > p$. Taking the $L^p(\Omega)$-norm and applying Minkowski and Jensen inequalities we get

$$\begin{aligned}\|\sup_{y\in J_1} |W(y)|\|_{L^p} &\leq \|W(y_{0,1})\|_{L^p} + \sum_{m=1}^{\infty} \Big(\mathbb{E} \Big(\sum_{i=1}^{2^{m(n-1)}} |W(y_{m,i}) - W(\pi(y_{m,i}))|^s \Big)^{p/s} \Big)^{1/p} \\ &\leq \|W(y_{0,1})\|_{L^p} + \sum_{m=1}^{\infty} \Big(\sum_{i=1}^{2^{m(n-1)}} \mathbb{E}\, |W(y_{m,i}) - W(\pi(y_{m,i}))|^s \Big)^{1/s}.\end{aligned}$$

The 1st term can be bounded from above by using (5) and Lemma 6, yielding $\|W(y_{0,1})\|_{L^p} \leq \widetilde{C}_p 2^l |t - t'|$, where $\widetilde{C}_p > 0$ is a constant. We split the remaining series into two parts, depending on whether 2^{-m-l} is smaller or larger than $|t - t'|$. For the tail when 2^{-m-l} is smaller we use the estimate

$$\begin{aligned}\mathbb{E}\, |W(y_{m,i}) - W(\pi(y_{m,i}))|^s &\leq 2^s \mathbb{E}\, |Z_l(y_{m,i}, t, x) - Z_l(\pi(y_{m,i}), t, x)|^s \\ &\quad + 2^s \mathbb{E}\, |Z_l(y_{m,i}, t', x) - Z_l(\pi(y_{m,i}), t', x)|^s \\ &\leq C_s 2^{s+1+ls-(m+l)s} \leq \widetilde{C}_s 2^{-ms},\end{aligned}$$

while for $|t - t'|$ smaller than 2^{-m-l} we use the estimate

$$\begin{aligned}\mathbb{E}\, |W(y_{m,i}) - W(\pi(y_{m,i}))|^s &\leq 2^s \mathbb{E}\, |Z_l(y_{m,i}, t, x) - Z_l(y_{m,i}, t', x)|^s \\ &\quad + 2^s \mathbb{E}\, |Z_l(\pi(y_{m,i}), t, x) - Z_l(\pi(y_{m,i}), t', x)|^s \\ &\leq C_s 2^{s+1+ls} |t - t'|^s \leq \widetilde{C}_s 2^{ls} |t - t'|^s.\end{aligned}$$

Here, $\widetilde{C}_s > 0$ is some constant. We thus get for $s > n-1$ that

$$\begin{aligned}\|\sup_{y \in J_1} |W(y)|\|_{L^r} &\le C_r^{1/r} 2^l |t-t'| + \sum_{m=1}^{\left\lfloor \log_2 \frac{1}{|t-t'|} \right\rfloor - l} (\widetilde{C}_s 2^{m(n-1)+ls} |t-t'|^s)^{1/s} \\ &\quad + \sum_{m=\left\lfloor \log_2 \frac{1}{|t-t'|} \right\rfloor - l}^{\infty} (\widetilde{C}_s 2^{m(n-1-s)})^{1/s} \\ &\le C 2^l |t-t'|^{1-\frac{n-1}{s}}\end{aligned}$$

for some constant $C > 0$. This is enough since we can choose s so large that $\frac{n-1}{s} r < \varepsilon$. ■

Proof of Proposition 12. We may assume that $p > 1$ is so small that $\alpha - p \operatorname{Re} \beta > 0$. Then

$$|E_l(x)|^p \lesssim e^{p(\operatorname{Re}\beta) Z_l(x) - p \frac{(\operatorname{Re}\beta)^2 - (\operatorname{Im}\beta)^2}{2} \mathbb{E} Z_l(x)^2}$$

and

$$\mathbf{1}_{A_l(x)} \le e^{(\alpha - p \operatorname{Re}\beta) Z_l(x) - (\alpha - p \operatorname{Re}\beta)\alpha \mathbb{E} Z_l(x)^2},$$

so

$$\sup_{x \in J} |E_l(x)|^p \mathbf{1}_{A_l(x)} \lesssim e^{\alpha \sup_{x \in J} Z_l(x) - p \frac{(\operatorname{Re}\beta)^2 - (\operatorname{Im}\beta)^2}{2} \mathbb{E} Z_l(x)^2 - (\alpha - p \operatorname{Re}\beta)\alpha \mathbb{E} Z_l(x)^2}.$$

By Lemma 13 we thus get for any $u > 1$ that

$$\mathbb{E} \sup_{x \in J} |E_l(x)|^p \mathbf{1}_{A_l(x)} \lesssim e^{\frac{u\alpha^2}{2} \log(2) l - p \frac{(\operatorname{Re}\beta)^2 - (\operatorname{Im}\beta)^2}{2} \mathbb{E} Z_l(x)^2 - (\alpha - p \operatorname{Re}\beta)\alpha \mathbb{E} Z_l(x)^2},$$

and by choosing u and p sufficiently close to 1, we may make the exponent as close to

$$\left[-\frac{(\alpha - \operatorname{Re}\beta)^2}{2} + \frac{(\operatorname{Im}\beta)^2}{2} \right] \log(2) l$$

as we wish, which proves the claim. ■

Having proved all the auxiliary results we need, we will now finish with the proof of Theorem 5. Let $n \ge 1$ and set $N = t_n - 1$ (recall that t_n was defined in (9)). We may then write $\mu_N(f;\beta)$ as the sum

$$\mu_N(f;\beta) = \sum_{l=0}^{n} \int_I f(x) E_n(x) \mathbf{1}_{A_l(x)} \mathbf{1}_{B_{l,n}(x)} \, dx.$$

Here the lth term of the sum contains the contribution from those points for which the last time the field is exceptionally large is l. By Minkowski's and Jensen's inequalities it follows that for $p \in (1,2)$ we have

$$\begin{aligned}
&\|\mu_N(f;\beta)\|_{L^p(\Omega)} \\
&\leq \sum_{l=0}^{n} \sum_{i=0}^{2^{d(l+c)}} \left\| \int_{I_{l,i}} f(x) E_n(x) \mathbf{1}_{A_l(x)} \mathbf{1}_{B_{l,n}(x)} \, \mathrm{d}x \right\|_{L^p(\Omega)} \\
&\leq \sum_{l=0}^{n} \sum_{i=0}^{2^{d(l+c)}} \left(\mathbb{E}\left[\left(\mathbb{E}\left[\left| \int_{I_{l,i}} f(x) E_n(x) \mathbf{1}_{A_l(x)} \mathbf{1}_{B_{l,n}(x)} \, \mathrm{d}x \right|^2 \middle| \mathcal{F}_l \right] \right)^{p/2} \right] \right)^{1/p},
\end{aligned}$$

where $c > 0$ is a constant depending on d, and for fixed l the sets $I_{l,i} \subset I$ are dyadic subcubes of I with side length 2^{-l-c}. (The reason we do not take the side length of the cubes to be exactly 2^{-l} here is because the assumption in Proposition 11 requires their diameter to be at most 2^{-l-1}. It is enough to take $c = \lceil \log_2(2\sqrt{d}) \rceil$.) By Propositions 11 and 12 we obtain for small enough p that

$$\begin{aligned}
\|\mu_N(f;\beta)\|_{L^p(\Omega)} &\lesssim \|f\|_\infty \sum_{l=0}^{n} \sum_{i=0}^{c2^{dl}} 2^{-dl} \big(\mathbb{E}\,[\sup_{x \in I_{l,i}} |E_l(x)|^p \mathbf{1}_{A_l(x)}] \big)^{1/p} \\
&\lesssim \|f\|_\infty \sum_{l=0}^{n} \sum_{i=0}^{c2^{dl}} 2^{-dl} e^{-\varepsilon l/p} \lesssim \|f\|_\infty .
\end{aligned}$$

Thus, we have proven the L^p boundedness along the subsequence $\mu_{t_n - 1}$, and since the L^p norms of a martingale are increasing, it follows that the whole sequence μ_n has bounded L^p norms. It is clear that the $p > 1$ for which we get the bound depends continuously on β. We obtain thus an open set U containing $(0, \sqrt{2d})$ such that for any compact $K \subset U$ there exists $p > 1$ for which we have the uniform $L^p(\Omega)$-boundedness of $\mu_n(f;\beta)$, $\beta \in K$.

To get rid of the assumption that $\delta > \sqrt{d}$, we can partition I into a finite number of dyadic cubes I_k, $1 \leq k \leq m$, with diameter less than δ. By Minkowski's inequality we have

$$\|\mu_n(f;\beta)\|_{L^p(\Omega)} \leq \sum_{k=1}^{m} \|\mu_n(f\chi_{I_k};\beta)\|_{L^p(\Omega)}$$

and the above proof works for every summand, which yields the result in the case of general δ.

Finally, we will prove the analyticity in β by considering $\mu_n(f;\cdot)$ as an element of a suitable function space possessing the Radon–Nikodym property. In the context of multiplicative chaos theory such considerations have previously appeared in [2, 12]. Let $B_{2r} = B(\beta_0, 2r) \subset U$ be an open ball of radius $2r > 0$ and let $p > 1$ be such that $\mathbb{E}\,|\mu_n(f;z)|^p$ is uniformly bounded both in $z \in B_r$ and $n \geq 1$. Then almost surely for all $n \geq 1$ the function $z \mapsto \mu_n(f;z)$ belongs to the Bergman space $A^p(B_r) = L^p(B_r) \cap H(B_r)$, where $H(B_r)$ is the space of analytic functions on B_r. As a closed subspace of $L^p(B_r)$ the space $A^p(B_r)$ has the Radon–Nikodym property [8, Theorem 1.3.18]. Hence, the uniform boundedness

$$\mathbb{E}\,\|\mu_n(f;\cdot)\|_{A^p(B_r)}^p = \int_{B_r} \mathbb{E}\,|\mu_n(f;x+iy)|^p \,\mathrm{d}x\,\mathrm{d}y \leq |B_r| \sup_{n\geq 1}\sup_{z\in B_r} \mathbb{E}\,|\mu_n(f;z)|^p < \infty$$

for $n \geq 1$ of the Bergman norm in $L^p(\Omega)$ implies that the $A^p(B_r)$-valued martingale $\mu_n(f;z)$ converges almost surely in $A^p(B_r)$. (See for instance [8, Theorem 3.3.16].) In particular we have uniform convergence on all compact subsets of B_r to an analytic function $z \mapsto \mu(f;z)$. This finishes the proof of Theorem 5.

As indicated in Remark 2, it is rather straightforward to improve the convergence $\mu_n(f;\beta) \to \mu(f;\beta)$ for fixed test functions f to convergence in a suitable negative-index Sobolev space.

Lemma 14. Let $W^s(\mathbf{T}^d)$ be the L^2-Sobolev space on the d-dimensional torus, which we identify with $[0,1)^d$. Then the sequence μ_n converges in $W^s(\mathbf{T}^d)$ for all $s < -d$.

Proof. The norm in $W^s(\mathbf{T}^d)$ is given by

$$\|\varphi\|_{W^s(\mathbf{T}^d)} = \sqrt{\sum_{k\in\mathbf{Z}^d} (1+|k|^2)^s |\widehat{\varphi}(k)|^2},$$

where

$$\widehat{\varphi}(k) = \int_I \varphi(x) e^{-2\pi i k\cdot x}\,\mathrm{d}x.$$

In particular we have

$$\mathbb{E}\,\|\mu_n\|_{W^s(\mathbf{T}^d)}^p \leq \sum_{k\in\mathbf{Z}^d} (1+|k|^2)^{sp/2}\mathbb{E}\,|\widehat{\mu}_n(k)|^p \lesssim \sum_{k\in\mathbf{Z}^d} (1+|k|^2)^{sp/2},$$

where $1 < p < 2$ is given by Theorem 5 and the 1st inequality follows from the subadditivity of $x \mapsto x^{p/2}$. If we take $s < -d$, we see that the martingale μ_n is bounded in $L^p(\Omega; W^s(\mathbf{T}^d))$. Because $W^s(\mathbf{T}^d)$ is reflexive, it has the Radon–Nikodym property [8, Theorem 1.3.21], and hence μ_n converges in $W^s(\mathbf{T}^d)$. ■

We will close this section by giving a short argument that in the case β is real and greater than $\sqrt{2d}$, the resulting measure is almost surely degenerate.

Lemma 15. Let $\beta > \sqrt{2d}$ be real. Then almost surely $\mu(I;\beta) = 0$.

Proof. It is enough to show that $\lim_{l\to\infty} \mathbb{E}\,|\int_I E_l(x)\,\mathrm{d}x|^p = 0$ for some $p < 1$. Let $I_{l,j}$, $j = 1, 2, \ldots, 2^{ld}$ be a dyadic partition of I into subcubes of side length 2^{-l}. We have by the subadditivity of the map $t \mapsto t^p$ that

$$\mathbb{E}\Big|\int_I E_l(x)\,\mathrm{d}x\Big|^p = \mathbb{E}\Big|\sum_{j=1}^{2^{ld}} \int_{I_{l,j}} E_l(x)\,\mathrm{d}x\Big|^p \le \sum_{j=1}^{2^{ld}} \mathbb{E}\Big|\int_{I_{l,j}} E_l(x)\,\mathrm{d}x\Big|^p \le \sum_{j=1}^{2^{ld}} 2^{-ldp}\mathbb{E}\sup_{x\in I_{l,j}} |E_l(x)|^p.$$

Using Lemma 13 we see that this is less than some constant times

$$\sum_{j=1}^{2^{ld}} 2^{-ldp} e^{-p\frac{\beta^2}{2}\log(2)l} e^{u\frac{p^2\beta^2}{2}\log(2)l} = 2^{l(d(1-p)-p\frac{\beta^2}{2}+u\frac{p^2\beta^2}{2})},$$

where $u > 1$ can be chosen arbitrarily close to 1. Let $p = 1-\varepsilon$ and $u = 1+\varepsilon^2$ and define

$$h(\varepsilon) = d(1-p) - p\frac{\beta^2}{2} + u\frac{p^2\beta^2}{2} = \frac{\beta^2}{2}\varepsilon^4 - \beta^2\varepsilon^3 + \beta^2\varepsilon^2 - \big(\frac{\beta^2}{2} - d\big)\varepsilon.$$

We are done if we can find $\varepsilon > 0$ such that $h(\varepsilon) < 0$, and such an ε exists because $h(0) = 0$ and $h'(0) = -\frac{\beta^2}{2} + d < 0$. ■

3 Examples

In this section we consider two basic examples of non-Gaussian chaos measures to which our results apply. In the end we also discuss some open questions.

3.1 The non-Gaussian Fourier series

As our 1st application of Theorem 5 we will prove Theorem 1. Recall that we are interested in the random Fourier series

$$\sum_{k=1}^{\infty} X_k(x) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}(A_k\cos(2\pi kx) + B_k\sin(2\pi kx)),$$

where A_k and B_k are i.i.d. random variables satisfying $\mathbb{E}\,e^{\lambda A_1} < \infty$ for all $\lambda \in \mathbf{R}$.

The assumptions of Theorem 5 are fairly straightforward to establish.

Proof of Theorem 1. The moment condition (4) is clear. The derivative of X_k in this case is

$$X_k'(x) = 2\pi\sqrt{k}(-A_k \sin(2\pi kx) + B_k \cos(2\pi kx)),$$

and it is easy to check that its supremum is

$$2\pi\sqrt{k}\sqrt{A_k^2 + B_k^2},$$

whose rth moment is of order $k^{r/2}$. Hence, there exists a constant $C > 0$ such that

$$\sum_{k=1}^{n} \mathbb{E}\,|\sup_{x\in I} X_k'(x)|^r \le Cn^{r/2+1}.$$

On the other hand, $e^{r\sum_{k=1}^{n}\mathbb{E}\,X_k(0)^2} = e^{r\log(n)+O(1)}$ is of order n^r so (7) clearly holds.

It remains to verify that the field has a locally log-correlated structure. The condition $\mathbb{E}\,X_k(x)^2 \to 0$ obviously holds. In (3) we choose $\delta = \frac{1}{2}$. Notice that $\sum_{k=1}^{n}\mathbb{E}\,X_k(0)^2 = \log(n) + O(1)$. Assume first that $|x-y| \le 1/n$. Then

$$\min(\log\frac{1}{|x-y|}, \sum_{k=1}^{n}\mathbb{E}\,X_k(0)^2) = \log(n) + O(1).$$

On the other hand,

$$\begin{aligned}\sum_{k=1}^{n}\mathbb{E}\,X_k(x)X_k(y) &= \sum_{k=1}^{n}\frac{\cos(2\pi k(x-y))}{k}\\ &= \sum_{k=1}^{n}\frac{1+O(k^2(x-y)^2)}{k} = \log(n) + O(1),\end{aligned}$$

which shows that (3) holds in this case. Assume then that $\frac{1}{2} \ge |x-y| > 1/n$. In this case

$$\begin{aligned}&\min(\log\frac{1}{|x-y|}, \sum_{k=1}^{n}\mathbb{E}\,X_k(0)^2) = \log\frac{1}{|x-y|} + O(1) = \log\frac{1}{2\sin(\pi|x-y|)} + O(1)\\ &= \sum_{k=1}^{n}\frac{\cos(2\pi k(x-y))}{k} + \sum_{k=n+1}^{\infty}\frac{\cos(2\pi k(x-y))}{k} + O(1),\end{aligned}$$

and it is enough to show that

$$\sum_{k=n+1}^{\infty}\frac{\cos(2\pi k(x-y))}{k}$$

is bounded. Let

$$R_n(t) = \sum_{k=n+1}^{\infty} \frac{\cos(2\pi kt)}{k} = \mathrm{Re} \sum_{k=n+1}^{\infty} \frac{e^{2\pi ikt}}{k}.$$

For all $N \geq n+1$ we have

$$\sum_{k=n+1}^{N} \frac{e^{2\pi ikt}}{k} = \sum_{k=n+1}^{N} e^{2\pi ikt} \int_0^1 r^{k-1}\,\mathrm{d}r = \int_0^1 \frac{e^{2\pi i(N+1)t} r^N - e^{2\pi i(n+1)t} r^n}{e^{2\pi it} r - 1}\,\mathrm{d}r,$$

and since we have the inequalities

$$|e^{2\pi it} r - 1| \geq \begin{cases} 1, & \text{when } \frac{1}{2} \geq |t| \geq \frac{1}{4}, \\ \sin(2\pi t), & \text{when } |t| \leq \frac{1}{4}, \end{cases}$$

we see that for fixed $|t| > 0$ the denominator is bounded and hence we may take limit as $N \to \infty$ to get

$$|R_n(t)| \leq \int_0^1 \frac{r^n}{|e^{2\pi it} r - 1|}\,\mathrm{d}r.$$

If $\frac{1}{2} \geq |t| \geq \frac{1}{4}$, we have $|R_n(t)| \leq \int_0^1 r^n\,\mathrm{d}r = \frac{1}{n+1}$ and similarly if $0 < |t| \leq \frac{1}{4}$, we have $|R_n(t)| \leq \frac{1}{(n+1)\sin(2\pi t)}$. This shows the boundedness of $|R_n(x-y)|$ when $\frac{1}{2} \geq |x-y| \geq \frac{1}{n}$. ■

3.2 Chaos induced by dilations of a stationary process

Our 2nd example is a small variation of the one given by Mannersalo, Norros, and Riedi in [15]. In their paper random measures of the form

$$\prod \Lambda_k(b^k x)\,\mathrm{d}x$$

were studied on the real line. Here Λ_k are nonnegative stationary i.i.d. random processes with expectation 1, $\mathrm{d}x$ is the one-dimensional Lebesgue measure, and $b > 1$ is a constant.

In the present example we define

$$X_k(x) = a_k Y_k(b_k x), \tag{20}$$

where Y_k are i.i.d. continuous, centered, and stationary random fields on $\mathbf{R}^d$ with the covariance function

$$\mathbb{E}\, Y_k(x) Y_k(y) = f(|x-y|)$$

and $a_k, b_k > 0$ are constants. Thus, the case $a_k = 1$, $b_k = b^k$ would allow us to write $\Lambda_k(x) = \frac{e^{Y_k(b^k x)}}{\mathbb{E}\, e^{Y_k(b^k x)}}$ and obtain the situation of [15]. We are not, however, quite able to consider this particular case since our method requires weak decay from the coefficients a_k.

First of all, we assume that

$$\mathbb{E}\, e^{\lambda Y_k(0)} < \infty \tag{21}$$

for all $\lambda \in \mathbf{R}$ and that $Y_k(x)$ satisfies the regularity condition

$$\mathbb{E}\, |Y_k(x) - Y_k(y)|^r \leq C_r |x-y|^r \quad \text{for all } r \geq 2, \tag{22}$$

where $C_r > 0$ is a constant that may depend on r. Furthermore, we require that for all $t > 0$ we have

$$f(t) = 1 + O(t) \quad \text{and} \quad f(t) = O(t^{-\delta}) \tag{23}$$

for some $\delta > 0$.

As a side remark, notice that if (22) holds for some sequence $r_n \to \infty$, then by Hölder's inequality it holds for all $r \geq 2$.

For this model we obtain the following result.

Theorem 16. Let $a_k > 0$ be such that

$$\lim_{k\to\infty} a_k = 0, \quad \sum_{k=1}^{\infty} a_k^2 = \infty, \quad \text{and} \quad \sum_{k=1}^{\infty} a_k^3 < \infty.$$

Assume that b_k are of the form

$$b_k = \exp(c_k + \sum_{j=1}^{k} a_j^2),$$

where $c_k \in \mathbf{R}$ satisfy $\sup_{k\geq 1} |c_k| < \infty$. Then the conclusions of Theorem 1 hold for the chaos measures

$$\mathrm{d}\mu_n = \frac{e^{\sum_{k=1}^n a_k Y_k(b_k x)}}{\mathbb{E}\, e^{\sum_{k=1}^n a_k Y_k(b_k x)}}\, \mathrm{d}x,$$

where Y_k is assumed to satisfy (21), (22), and (23).

Toward the proof of the above result, we define

$$A_n = \sum_{k=1}^{n} \mathbb{E}\, X_k(0)^2 = \sum_{k=1}^{n} a_k^2, \quad n \geq 1.$$

We then have the following lemma.

Lemma 17. Let $\alpha \in \mathbf{R}$, $\alpha \neq 0$. Then there exists $C_\alpha > 0$ such that

$$\sum_{k=m}^{n} a_k^2 e^{\alpha A_k} \leq C_\alpha |e^{\alpha A_n} - e^{\alpha A_{m-1}}|$$

for all $n \geq m \geq 1$.

Proof. Assume first that $\alpha > 0$. We have

$$\sum_{k=m}^{n} a_k^2 e^{\alpha A_k} = \sum_{k=m}^{n} (A_k - A_{k-1}) e^{\alpha A_k}.$$

Let $S = \sup_{k \geq 1} a_k^2$. Since $a_k \to 0$, there exists $k_0 \geq 1$ such that for $k \geq k_0$ we have $A_k - A_{k-1} \leq 1$. The elementary inequality $x \leq \frac{S}{1-e^{-\alpha S}}(1 - e^{-\alpha x})$ for $0 \leq x \leq S$ gives us

$$\sum_{k=m}^{n} a_k^2 e^{\alpha A_k} \leq C_\alpha \sum_{k=m}^{n} (e^{\alpha A_k} - e^{\alpha A_{k-1}}) = C_\alpha (e^{\alpha A_n} - e^{\alpha A_{m-1}}),$$

where $C_\alpha = \frac{S}{1-e^{-\alpha S}}$.

On the other hand, if $\alpha < 0$, then we have the inequality $x \leq \frac{-1}{\alpha}(e^{-\alpha x} - 1)$ for $x \geq 0$. Thus,

$$\sum_{k=m}^{n} a_k^2 e^{\alpha A_k} \leq C_\alpha \sum_{k=m}^{n} (e^{\alpha A_{k-1}} - e^{\alpha A_k}) = C_\alpha (e^{\alpha A_{m-1}} - e^{\alpha A_n}),$$

where $C_\alpha = \frac{-1}{\alpha}$. ■

Proof of Theorem 16. Once again, it is enough to check the assumptions of Theorem 5. We will start by showing that we have a locally log-correlated structure. For this it is enough to verify that (3) holds, the other conditions being trivial. Assume first that $n \geq 1$ and $x, y \in I$ are such that $A_n < \log \frac{1}{|x-y|}$ holds. Then by Lemma 17 the left-hand side of (3) satisfies the inequality

$$\begin{aligned}\Big| \sum_{k=1}^{n} \mathbb{E} X_k(x) X_k(y) - A_n \Big| &\leq \sum_{k=1}^{n} a_k^2 |f(b_k|x-y|) - 1| \lesssim \sum_{k=1}^{n} a_k^2 b_k |x-y| \\ &\lesssim \sum_{k=1}^{n} a_k^2 e^{A_k} |x-y| \lesssim (1 + e^{A_n})|x-y| \lesssim 1. \end{aligned} \tag{24}$$

Thus, (3) holds in this case. Assume then that $A_n \geq \log \frac{1}{|x-y|}$ and let N be the last index for which $A_N < \log \frac{1}{|x-y|}$. In this case we have

$$\sum_{k=1}^{n} \mathbb{E}\, X_k(x) X_k(y) - \log \frac{1}{|x-y|} = [\sum_{k=1}^{N} a_k^2 f(b_k|x-y|) - A_N] \\ + [A_N - \log \frac{1}{|x-y|}] + \sum_{k=N+1}^{n} a_k^2 f(b_k|x-y|).$$

The 1st term on the right-hand side is bounded by (24) and the 2nd term is bounded simply by the way A_N was chosen. Finally, again by Lemma 17, we have

$$\sum_{k=N+1}^{n} a_k^2 |f(b_k|x-y|)| \lesssim \sum_{k=N+1}^{n} a_k^2 b_k^{-\delta} |x-y|^{-\delta} \lesssim |x-y|^{-\delta} e^{-\delta A_N}.$$

Since $e^{-\delta A_N} \approx |x-y|^{\delta}$, we see that (3) holds also in this case.

Next we will verify the regularity conditions (4) and (5). Clearly, the requirement $\sup_{x \in I} \mathbb{E}\, e^{\lambda X_k(x)} < \infty$ holds by assumption (21). Moreover, we have

$$\left(\mathbb{E}\,|X_k(x)|^{3+\varepsilon}\right)^{\frac{3}{3+\varepsilon}} = a_k^3 \left(\mathbb{E}\,|Y_k(0)|^{3+\varepsilon}\right)^{\frac{3}{3+\varepsilon}},$$

and thus (4) holds. Finally, we have

$$\sum_{k=1}^{n} \mathbb{E}\,|X_k(x) - X_k(y)|^r \leq C_r \sum_{k=1}^{n} a_k^r b_k^r |x-y|^r \lesssim C_r b_n^r |x-y|^r \lesssim C_r e^{r \sum_{k=1}^{n} a_k^2} |x-y|^r$$

by Lemma 17, so (6) holds and therefore also the condition (5) follows. ■

To illustrate the relationship between a_k and b_k in Theorem 16, we mention the following corollary.

Corollary 18. Theorem 16 holds in the following three cases:

- $a_k = k^{-\alpha}$ for some $\frac{1}{3} < \alpha < \frac{1}{2}$ and $b_k = e^{\frac{k^{1-2\alpha}}{1-2\alpha}}$,
- $a_k = k^{-1/2}$ and $b_k = k$, or
- $a_k = (k \log k)^{-1/2}$ and $b_k = \log k$ when $k \geq 2$ and $a_1 = b_1 = 0$.

As an example of stationary process for which our theorem is valid one can take a stationary Gaussian process $Y(x)$ whose covariance function f satisfies $f(t) = 1 + O(t^2)$

and decays like $t^{-\delta}$ for some $\delta > 0$. Indeed, in this case we have

$$\mathbb{E}\,|Y(x) - Y(y)|^r \lesssim_r (\mathbb{E}\,|Y(x) - Y(y)|^2)^{\frac{r}{2}} = 2^{\frac{r}{2}}(1 - f(|x-y|))^{\frac{r}{2}} \lesssim_r |x-y|^r.$$

A very simple non-Gaussian example is given by $Y(x) = A\cos(x + U)$, where A and U are independent, U is uniformly distributed in $[0, 2\pi)$, $\mathbb{E}\,A = 0$, $\mathbb{E}\,A^2 = 2$, and $\mathbb{E}\,e^{\lambda A} < \infty$ for all $\lambda \in \mathbf{R}$. It is easy to construct more complicated families of non-Gaussian stationary processes satisfying the conditions of the theorem.

3.3 Open questions

On the foundational level several questions are widely open in the case of non-Gaussian chaos. An important property of Gaussian chaos is universality, that is, independence of the chaos of the approximation used. For results in this direction, see, for example [9, 19, 22]. No such general results are known in the non-Gaussian case. Another interesting open problem is the construction of critical chaoses.

Moreover, it would be interesting to study the finer properties of the resulting chaoses. For example, can one determine the multifractal spectrum of the measure? Is it possible to determine the exact L^p-integrability or the tail behavior of the total mass? In general one may try to examine what are the differences and similarities of these measures to their Gaussian counterparts.

Funding

This work was supported by the Doctoral Programme in Mathematics and Statistics at University of Helsinki, the Academy of Finland CoE 'Analysis and Dynamics', as well as the Academy of Finland Project 'Conformal methods in analysis and random geometry'.

Acknowledgments

I wish to thank my thesis advisor Eero Saksman for proposing this problem, as well as for the many fruitful discussions regarding it and other topics. Many thanks also to Christian Webb and Julien Barral for their valuable comments on the paper.

References

[1] Bacry, E. and J. F. Muzy. "Log-infinitely divisible multifractal processes." *Comm. Math. Phys.* 236, no. 3 (2003): 449–75.

[2] Barral, J. "Generalized vector multiplicative cascades." *Adv. Appl. Probab.* 33, no. 4 (2001): 874–95.

[3] Barral, J. and B. Mandelbrot. "Multifractal products of cylindrical pulses." *Probab. Theory Relat. Fields* 124, no. 3 (2002): 409–30.

[4] Barral, J. and B. Mandelbrot. "Random Multiplicative Multifractal Measures, Part II." *In Fractal Geometry and Applications: A Jubilee of Benoıt Mandelbrot. Proceedings of Symposia in Pure Mathematics.* 2004. Providence, Rhode Island: American Mathematical Society.

[5] Berestycki, N. "An elementary approach to Gaussian multiplicative chaos." *Electron. Commun. Probab.* 22 (2017).

[6] Duplantier, B., R. Rhodes, S. Sheffield, and V. Vargas. "Critical Gaussian multiplicative chaos: convergence of the derivative martingale." *Ann. Probab.* 42, no. 5 (2014): 1769–808.

[7] Duplantier, B., R. Rhodes, S. Sheffield, and V. Vargas. "Renormalization of critical Gaussian multiplicative chaos and KPZ relation." *Comm. Math. Phys.* 330, no. 1 (2014): 283–330.

[8] Hytönen, T., J. Van Neerven, M. Veraar, and L. Weis. *Analysis in Banach Spaces. Volume I: Martingales and Littlewood Paley Theory*. Cham, Switzerland: Springer, 2016.

[9] Junnila, J. and E. Saksman. "Uniqueness of critical Gaussian chaos." *Electron. J. Probab.* 22 (2017).

[10] Kahane, J.-P. "Sur le chaos multiplicatif." *Ann. Math. Qué.* 9, no. 2 (1985): 105–50.

[11] Kahane, J.-P. "Positive martingales and random measures." *Chin. Ann. Math. Ser. B* 8, no. 1 (1987): 1–12.

[12] Kahane, J.-P. *Multiplicative Chaos and Multimeasures*, 115–26. *Complex Analysis, Operators and Related Topics*. Birkhäuser, 2000.

[13] Madaule, T., R. Rhodes, and V. Vargas. "Glassy phase and freezing of log-correlated Gaussian potentials." *Ann. Appl. Probab.* 26, no. 2 (2016): 643–90.

[14] Mandelbrot, B. "Multiplications aléatoires itérées et distributions invariantes par moyenne pondérée aléatoire." *C. R. Math. Acad. Sci. Paris* 278 (1974): 289–92 and 355-8.

[15] Mannersalo, P., I. Norros, and R. Riedi. "Multifractal products of stochastic processes: construction and some basic properties." *Adv. Appl. Probab.* 34, no. 4 (2002): 888–903.

[16] Rhodes, R., J. Sohier, and V. Vargas. "Levy multiplicative chaos and star scale invariant random measures." *Ann. Probab.* 42, no. 2 (2014): 689 –724.

[17] Rhodes, R. and V. Vargas. "Gaussian multiplicative chaos and applications: a review." *Probab. Surv.* 11 (2014): 315–392.

[18] Rhodes, R. and V. Vargas. "*Lecture notes on Gaussian multiplicative chaos and Liouville quantum gravity.*" (2016): preprint arXiv preprint 1602.07323.

[19] Robert, R. and V. Vargas. "Gaussian multiplicative chaos revisited." *Ann. Probab.* 38, no. 2 (2010): 605–31.

[20] Rosenthal, H. "On the subspaces of $L^p (p > 2)$ spanned by sequences of independent random variables." *Israel J. Math.* 8, no. 3 (1970): 273–303.

[21] Saksman, E. and C. Webb. "*Multiplicative chaos measures for a random model of the Riemann zeta function*." (2016): preprint arXiv preprint 1604.08378.

[22] Shamov, A. "On Gaussian multiplicative chaos." *J. Funct. Anal.* 270, no. 9 (2016): 3224 –61.

ARTICLE III

J. Junnila, E. Saksman, and C. Webb

Imaginary multiplicative chaos: Moments, regularity and connections to the Ising model

To be submitted (*2018*)

III

IMAGINARY MULTIPLICATIVE CHAOS: MOMENTS, REGULARITY AND CONNECTIONS TO THE ISING MODEL

JANNE JUNNILA, EERO SAKSMAN, AND CHRISTIAN WEBB

Abstract. In this article we study imaginary Gaussian multiplicative chaos – namely a family of random generalized functions which can formally be written as $e^{iX(x)}$, where X is a log-correlated real-valued Gaussian field on $\mathbf{R}^d$, i.e. it has a logarithmic singularity on the diagonal of its covariance. We study basic analytic properties of these random generalized functions, such as what spaces of distributions do these objects live in, along with their basic stochastic properties, such as moment and tail estimates.

After this, we discuss connections between imaginary multiplicative chaos and the critical planar Ising model, namely that the scaling limit of the spin field of the so called critical planar XOR-Ising model can be expressed in terms of the cosine of the Gaussian free field, i.e. the real part of an imaginary multiplicative chaos distribution. Moreover, if one adds a magnetic perturbation to the XOR-Ising model, then the scaling limit of the spin field can be expressed in terms of the cosine of the sine-Gordon field, which can also be viewed as the real part of an imaginary multiplicative chaos distribution.

The first sections of the article have been written in the style of a review, and we hope that the text will also serve as an introduction to imaginary chaos for an uninitiated reader.

1. Introduction

We begin this introduction with Section 1.1, where we informally review what log-correlated fields and multiplicative chaos are as well as their role in modern probability theory and applications. Then in Section 1.2, we state our main results concerning the existence and basic properties of imaginary multiplicative chaos. After this, we move to Section 1.3, where we discuss our results concerning the Ising model and Section 1.4, where we discuss some applications in random matrix theory. Finally in Section 1.5, we give an outline of the remainder of the article.

1.1. **Background on log-correlated fields and multiplicative chaos.** Log-correlated fields – namely real-valued random generalized functions on $\mathbf{R}^d$ with a logarithmic singularity on the diagonal of the covariance kernel[1] – have emerged as an important class of objects playing a central role in various probabilistic models. For example, one encounters them when studying the statistical behavior of the Riemann zeta function on the critical line [2, 40, 68, 75], characteristic polynomials of large random matrices [47, 41, 74], combinatorial models for random partitions of integers [50], certain models of mathematical finance [73, Section 5], lattice models of statistical mechanics [55], construction of conformally invariant random planar curves such as Stochastic Loewner evolution [3, 77], the random geometry of two-dimensional quantum gravity and scaling limits of random planar maps [22, 56, 32, 67], growth models [12], and statistical properties of disordered systems [17]. A typical example of a log-correlated field is the two-dimensional Gaussian free field, namely the centered Gaussian process on a planar domain with covariance given by the Green's function of the domain with some prescribed boundary conditions. In the planar case, a log-correlated field can be seen as a model for a generic random surface.

In many of the above cases, a central goal is to understand geometric properties of the object described in terms of the log-correlated field. One might for example be interested in understanding how the maximum of the field behaves or one might be interested in the Hausdorff dimensions of level sets of the field. As the field is a rough object – a random generalized function instead of a random function – it is not obvious that any of these notions make sense. Nevertheless, in some specific situations, precise sense can be made of such questions – for such studies, see e.g. [1, 2, 17, 19, 29, 39,

[1] For precise definitions in the Gaussian setting, see Section 2.

40, 58, 59, 65, 68, 72]. In some approaches to such geometric questions, an important role is played by a family of random measures which can be formally written as the exponential of the log-correlated field multiplied by a real parameter: $e^{\beta X(x)}dx$, where X is the log-correlated field and $\beta > 0$. The rigorous construction of these measures requires a regularization and renormalization procedure since a priori, one can not exponentiate a generalized function. The theory of these random measures goes under the name of multiplicative chaos and its foundations were laid by Kahane [53]; see also [73] for a recent review of the theory and [9] for an elegant and concise construction of the family of measures. The connection between multiplicative chaos and the geometry of the field can be seen e.g. in [73, Section 4] or the approach of [9]. In addition to being of importance in geometric studies, multiplicative chaos measures also a play an important role in a rigorous definition of so-called Liouville field theory – an example of a quantum field theory with certain symmetries under conformal transformations; for details, see e.g. [22, 56].

The importance of these multiplicative chaos measures suggests posing the question of whether one can make sense of similar objects for complex values of the parameter β in the definition of $e^{\beta X(x)}$, or more generally can one consider similar objects for complex log-correlated fields[2]. Moreover, if one can make sense such objects, what properties do they have, where do they arise, and do they perhaps say something about the geometry of the field X? Indeed, such objects have been studied – see e.g. [6, 57] – and also show up naturally when studying the statistics of the Riemann zeta function on the critical line and characteristic polynomials of random matrices – see [75]. We also point out that, at least on a formal level, the situation where the parameter β is purely imaginary plays a central role in the study of so-called imaginary geometry – see [66].

The purpose of this article is to study in more detail a particular case of such complex multiplicative chaos in that we consider the situation where the relevant parameter is purely imaginary: we consider objects formally written as $e^{i\beta X(x)}$, where $\beta \in \mathbf{R}$, and X is a real-valued Gaussian log-correlated field – imaginary Gaussian multiplicative chaos. We have two primary goals for this article. The first one is to study the basic properties of these objects as random generalized functions. Thus we investigate their analytic properties – namely show that the relevant objects exist as certain random generalized functions and study their smoothness properties – and also their basic probabilistic properties – namely provide moment and tail estimates for relevant quantities built from this imaginary chaos. In this latter part the main novelty of our results is that they deal with general log-correlated fields, contrary to previous studies dealing with the Gaussian free field, as it turns out that the general case requires new tools. Our second goal is to prove that imaginary multiplicative chaos is a class of probabilistic objects arising naturally e.g. in models of statistical mechanics.[3] In addition to these primary goals, we use an example from random matrix theory to illustrate that there are some subtleties in constructing multiplicative chaos, both in the real and imaginary case.

As we suspect that imaginary multiplicative chaos will play a prominent role in different types of probabilistic models, we have tried to write this article in a format similar to a survey article. In particular, we review basic properties of imaginary multiplicative chaos and discuss different types of results in a style which is hopefully accessible to readers of various backgrounds and interests.

We now turn to discussing more precisely our main results.

1.2. **Main results on basic properties of imaginary multiplicative chaos.**

Naturally the starting point in discussing basic properties of imaginary multiplicative chaos is the existence of imaginary multiplicative chaos. That is, given a centered Gaussian process X taking values in some space of generalized functions, and (formally) having a covariance kernel of the form

$$C_X(x,y) = \mathbb{E}\, X(x)X(y) = \log|x-y|^{-1} + g(x,y), \tag{1.1}$$

[2] That is random fields whose real and imaginary parts are real-valued log-correlated fields.

[3] On a possibly related issue, we remark that we suspect that as suggested in the theoretical physics literature, imaginary multiplicative chaos can be used to give a rigorous definition of the Coulomb gas formulation of some conformal field theories, though we do not discuss this further here.

where g is say locally bounded (see Section 2 for details), we want to make sense of $e^{i\beta X(x)}$ in some way. Some results of this flavor actually exist already – see e.g. [6, 57],[4] but there are many natural questions that remain unanswered about the objects. More precisely, [6, 57] impose some assumptions on the function g, that one would expect to be rather unnecessary, and based on their results, very little can be said about the analytic properties of the objects $e^{i\beta X(x)}$ – are they possibly random smooth functions, are they random complex measures, or are they random generalized functions? Also probabilistic properties such as precise tail estimates are not studied in [6, 57], though we do refer to [61, Appendix A], where such questions are studied in the setting of the Gaussian free field. Hence, one of our main goals is to address these issues, namely to study imaginary chaos for a rather general class of covariances C_X, to describe nearly optimal regularity results, as well as probabilistic results such a moment and tail estimates.

We now describe the setting of our first result concerning existence and uniqueness of imaginary chaos for rather general covariances C_X. As X is a random generalized function instead of an honest function, $e^{i\beta X(x)}$ can not be constructed in a naive way. Instead, one must construct it through a regularization and limiting procedure. More precisely, we introduce suitable approximations to X, which are honest functions and which we call standard approximations X_n – see Definition 2.7 for a precise definition. Standard approximations always exist – a typical example of a standard approximation is convolving X with a smooth bump function; see Lemma 2.8 for details. One would then expect that the correct way to construct $e^{i\beta X(x)}$ is as a limit of the sequence $e^{i\beta X_n(x)+\frac{\beta^2}{2}\mathbb{E}[X_n(x)^2]}$. This turns out to be partially true – as proven in [57] under some further assumptions on C_X and for a rather particular approximation, this sequence has a non-trivial limit for[5] $0<\beta<\sqrt{d}$. For larger β, it was shown in [57] that (once again under certain assumptions on C_X) that one can multiply $e^{i\beta X_n(x)}$ by a suitable deterministic n-dependent factor to obtain convergence to complex white noise. As white noise is a well understood object, we have chosen to focus on the regime $0<\beta<\sqrt{d}$ in this article. In addition to being able to construct $e^{i\beta X(x)}$ as a limit of $e^{i\beta X_n(x)+\frac{\beta^2}{2}\mathbb{E}\,X_n(x)^2}$, one might hope that the limiting object would not depend very much on how we approximated X – this is indeed confirmed by one of our results. Finally, as mentioned in the previous section, the limiting object is rather rough; a generalized function instead of an honest function so we formulate convergence in a suitable Sobolev space of generalized functions – we refer the reader wishing to recall the definition of the Sobolev space $H^s(\mathbf{R}^d)$ to the beginning of Section 2.2. The precise statement concerning all these issues is the following theorem.

Theorem 1.1. *Let $(X_n)_{n\geq 1}$ be a standard approximation of a given log-correlated field X on a domain $U\subset\mathbf{R}^d$ satisfying the assumptions* (2.1) *and* (2.2) (*see also Definition 2.7 for a precise definition of a standard approximation*). *When $0<\beta<\sqrt{d}$, the functions*

$$\mu_n(x)=e^{i\beta X_n(x)+\frac{\beta^2}{2}\mathbb{E}[X_n(x)^2]},$$

understood as zero outside of U, converge in probability in $H^s(\mathbf{R}^d)$ for $s<-\frac{d}{2}$. The limit μ is a non-trivial random element of $H^s(\mathbf{R}^d)$, supported on $\overline{U}$.

Moreover, suppose that X_n and $\widetilde{X}_n$ are two sequences of standard approximations of the same log-correlated field X (*satisfying assumptions* (2.1) *and* (2.2) *below*), *living on the same probability space, and satisfying*

$$\lim_{n\to\infty}\mathbb{E}\,X_n(x)\widetilde{X}_n(y)=C_X(x,y), \tag{1.2}$$

where convergence takes place in measure on $U\times U$. Then the corresponding imaginary chaoses μ and $\widetilde{\mu}$ are equal almost surely.

[4]In the setting of the Gaussian free field, a very similar question though with a different emphasis has been considered already in [37]; a study related to the sine-Gordon model – see Section 1.3 for further discussion about the sine-Gordon model and its relationship to imaginary chaos.

[5]Note that as we are dealing with a centered Gaussian field, $-X\stackrel{d}{=}X$ so results for $-\sqrt{d}<\beta<0$ can be obtained from the $0<\beta<\sqrt{d}$ case.

Our proof of this theorem, which does not rely on martingale theory as in [6, 57], is a rather basic probabilistic argument involving calculating second moments of objects such as $\int f(x)\mu_n(x)dx$ for suitable f – the proof is the main content of Section 3.1. We wish to point out here that one can show that different convolution approximations satisfy the condition (1.2) so this theorem shows that the limiting random variable μ is indeed unique at least if one restricts one's attention to convolution approximations.

As discussed earlier, one of our main goals in this article is to understand (essentially optimal) regularity properties of the object μ. While convergence in the space $H^s(\mathbf{R}^d)$ with $s < -\frac{d}{2}$ means that μ can not be terribly rough, it does not say that μ isn't say a C^∞-function. The following result, which is our main result concerning analytic properties of imaginary multiplicative chaos, rules out this kind of possibility, or even the possibility that μ would be a complex measure. As this means that μ is a true generalized function, we also study more extensively to which spaces of generalized functions does μ belong to and essentially extract its optimal regularity. For a reminder of the relevant spaces of generalized functions: $B^s_{p,q}$, Triebel spaces, etc., along with their uses, we refer the reader again to Section 2.2.

Theorem 1.2. *Let μ be the imaginary multiplicative chaos given by Theorem 1.1 and let $1 \le p, q \le \infty$. Then the following are true.*

(i) *μ is almost surely not a complex measure.*
(ii) *We have almost surely $\mu \in B^s_{p,q,loc}(U)$ when $s < -\frac{\beta^2}{2}$ and $\mu \notin B^s_{p,q,loc}(U)$ when $s > -\frac{\beta^2}{2}$.*
(iii) *Assume moreover that the function g from (1.1) satisfies $g \in L^\infty(U \times U)$ or that X is the GFF with zero boundary conditions – see Example 2.6. Then almost surely $\mu \in B^s_{p,q}(\mathbf{R}^d)$ when $s < -\frac{\beta^2}{2}$.*
(iv) *Analogous statements hold for Triebel spaces with $p, q \in [1, \infty)$.*

For obtaining upper bounds on regularity, our proof of this theorem relies on estimating low order moments of $\mu(f_k)$ for a suitable sequence of (random) test functions, while for lower bounds we combine the Fourier-analytic definition of Besov spaces with moment estimates of $\mu(f)$ for general deterministic f – the details of the proof are presented in Section 3.3.

Having described our main results concerning analytic properties of imaginary multiplicative chaos, we move onto basic probabilistic properties of it. The main question we wish to answer is what can be said about the law of $\mu(f) = \lim_{n\to\infty} \int e^{i\beta X_n(x) + \frac{\beta^2}{2}\mathbb{E}[X_n(x)^2]} f(x)dx$ for a given $f \in C_c^\infty(U)$. Our study of this question is through analysis of moments of $\mu(f)$. The existence of all positive moments is one of the main things that makes imaginary multiplicative chaos special compared to real or general complex chaos. More precisely, if one considers general complex multiplicative chaos, formally written as $e^{\beta X(x)}$ with $\mathrm{Re}(\beta) \neq 0$, then it is known that generically $\mathbb{E}\,|\int f(x)e^{\beta X(x)}|^k$ will be finite only for $0 \le k \le k_0$ for some finite k_0. We will show that for purely imaginary chaos, all moments exist. Moreover, as we will see, the moments grow slowly enough for the law of the random variable $\mu(f)$ to be characterized by the moments $\mathbb{E}\,\mu(f)^k\overline{\mu(f)}^l$, with k, l non-negative integers. A similar phenomenon has been observed for a particular model of what might be called signed multiplicative chaos – see [7].

The fact that the moments $\mu(f)$ grow slowly enough to determine the distribution for a particular variant of the Gaussian free field (corresponding to $g = 0$ in (1.1)) follows from the work in [45, 61]. Interesting related estimates in connection with the sine-Gordon model are obtained in [46]. However, the case of general g in (1.1) leads to surprising complications. Our analysis of moments is based on variants and generalizations of a famous inequality originally due to Onsager [71], that is often called Onsager's lemma (see e.g. [34]), or the electrostatic inequality (see e.g. [45]), as it involves the Green's function of the Laplacian in its original form. As we are not focusing on the Green's function, we find it more suitable to simply refer to our inequalities as Onsager (type) inequalities. As these Onsager inequalities are not directly properties of multiplicative chaos, we don't record them in this introduction, but refer the reader to Section 3.2 – see Proposition 3.6, Theorem 3.8, and Proposition 3.9. Of these results, we prove in Section 3.2 Proposition 3.6 and Proposition 3.9 which apply in the case $d = 2$. We prove Theorem 3.8, which applies for $d \neq 2$, in a separate article [52]. There the proof of Theorem 3.8 is based on a non-trivial decomposition result for log-correlated fields, which has several other applications as well, and we hence find it more suited for a separate publication. While one might argue that the most interesting log-correlated fields are variants of the Gaussian free field in two dimensions, we chose to present results for general d as there are natural one-dimensional log-correlated fields arising e.g. in random matrix theory [41, 47] and also four-dimensional ones arising in the study of the uniform spanning forest – see [60].

Given our Onsager inequalities, we may then deduce that all positive integer moments of imaginary multiplicative chaos are finite, and in fact grow slowly enough to determine the law of imaginary chaos – which can be seen as another kind of uniqueness result. More precisely, we have the following result, which requires some further regularity from the covariance of our log-correlated field.

Theorem 1.3. *Let X be a log-correlated field on $U \subset \mathbf{R}^d$ satisfying the conditions* (2.1) *and* (2.2)*. Let $0 < \beta < \sqrt{d}$ and μ be the random generalized function provided by Theorem 1.1. Then for $f \in C_c^\infty(U)$, $\mathbb{E}\,|\mu(f)|^k < \infty$ for all $k > 0$. For $d = 2$, assume further that the function g from* (2.1) *satisfies $g \in C^2(U \times U)$ and for $d \neq 2$, assume that $g \in H_{loc}^{d+\varepsilon}(U \times U)$ for some $\varepsilon > 0$.*[6] *Then there exists a constant $C > 0$ independent of f and N such that for $N \in \mathbf{Z}_+$*

$$\mathbb{E}\,|\mu(f)|^{2N} \leq \|f\|_\infty^{2N} C^N N^{\frac{\beta^2}{d}N}.$$

In particular, the law of $\mu(f)$ is determined by the moments $\mathbb{E}\,\mu(f)^k\overline{\mu(f)}^l$ with l, k non-negative integers and $\mathbb{E}\,e^{\lambda|\mu(f)|} < \infty$ for all $\lambda > 0$.

In the special case of $d = 2$, $f = 1$[7], and $g = 0$, such moments can in fact be interpreted as the canonical partition function of the so-called two-dimensional two-component plasma or neutral Coulomb gas. The connection between this model and imaginary multiplicative chaos was noted in [61, Appendix A], where using the main results of [61], very precise asymptotics for these moments were derived. Moreover, using these precise asymptotics, precise estimates for the tail of the distribution of the random variable one might formally write as $|\mu(1)|$ were derived. In this spirit, we combine Theorem 1.3 and Proposition 3.14 to obtain similar but slightly weaker results for general d, g, f:

Theorem 1.4. *Let X be a log-correlated field on U satisfying the conditions* (2.1) *and* (2.2)*. Assume further that the function g from* (2.1) *satisfies the following condition: if $d = 2$, $g \in C^2(U \times U)$ and if $d \neq 2$, $g \in H_{loc}^{d+\varepsilon}(U \times U)$ for some $\varepsilon > 0$. Now let $0 < \beta < \sqrt{d}$ and μ be the random generalized function from Theorem 1.1. Then for $f \in C_c^\infty(U)$*

$$\limsup_{\lambda \to \infty} \frac{\log \mathbb{P}(|\mu(f)| > \lambda)}{\lambda^{\frac{2d}{\beta^2}}} < 0.$$

Let us now assume further that $f \geq 0$ and f is not identically zero. Then for any $\varepsilon > 0$ we have

$$\liminf_{\lambda \to \infty} \frac{\log \mathbb{P}(|\mu(f)| > \lambda)}{\lambda^{\frac{2d}{\beta^2}+\varepsilon}} > -\infty.$$

We also point out that in [69, Proposition 17], similar tail bounds in the setting of the Gaussian free field (or more precisely, $g = 0$) were used to establish a Lee–Yang property for imaginary multiplicative chaos.

Proving the results discussed here is the main content of Section 3. In addition to these results, we also consider what we call universality properties of imaginary chaos in Section 3.4, where we show that through a similar regularization/renormalization scheme, one can make sense of $H(X)$ for a large class of periodic functions H, and the relevant object can be expressed in general in terms of imaginary multiplicative chaos – see Theorem 3.18. In Section 3.5, we study how the objects μ

[6] For the definition of the Sobolev space $H_{loc}^{d+\varepsilon}(U \times U)$, see Section 2.2.

[7] Note that we require test functions to have compact support so in our setting $f = 1$ is not strictly speaking a valid test function for μ, but if one were not interested in realizing μ as a random generalized function, one could simply consider the sequence of random variables $\mu_n(1)$, which are perfectly well defined, and show that these converge to something non-trivial. Such a phenomenon of being able to make sense of a random generalized function acting on a single test function which is not a priori a valid test function is common, and occurs e.g. for white noise.

behave in the vicinity of the critical point $\beta = \sqrt{d}$. More precisely, we prove in Theorem 3.20 that once one multiplies $\mu = \mu_\beta$ by a suitable deterministic quantity tending to zero as $\beta \nearrow \sqrt{d}$, one has convergence to a weighted complex white noise.

This concludes the summary of our results concerning the basic analytic and probabilistic properties of imaginary multiplicative chaos. We now turn to the connection between the Ising model and the random generalized functions μ of Theorem 1.1.

1.3. **Main results on the Ising model and multiplicative chaos.** In this section we review our basic results concerning the Ising model and imaginary chaos, beginning with some background to the problem we study. The Ising model is one of the most studied models of statistical mechanics, where the object of interest is a random spin configuration on some graph, or in other words, a random function defined on say the vertices of the graph and taking values ± 1. The model is known to describe certain aspects of ferromagnets – for its definition (in two dimensions and + boundary conditions), see Section 4.1 and for an extensive introduction to it, see e.g. [8]. A particularly important property of the Ising model on say $\mathbf{Z}^d$ with $d \geq 2$, is that at a certain temperature, known as the critical temperature, the model undergoes a phase transition, and the behavior of the correlation functions of the spin configuration change abruptly. It has been recently proven in [18] that for $d = 2$ and precisely at this critical temperature, these correlation functions have a non-trivial scaling limit and this scaling limit possesses certain conformal symmetries – see Theorem 4.1 where we recall this result. Indeed, physicists know that quite generically, models of statistical physics at their critical points[8] have scaling limits which can be described by quantum field theories behaving nicely under conformal transformations. While rigorously proving such statements has turned out to be very challenging for mathematicians, there has been rather spectacular progress in this direction in the case of the two-dimensional Ising model over the past two decades.

A particularly successful method for making precise mathematical sense of quantum field theories has been constructing probability measures on suitable spaces of generalized functions and proving that the relevant quantum field theory can be constructed from these random generalized functions – we refer the interested reader to [43] for further details about this construction. This kind of procedure has in fact more or less been carried out for the critical planar Ising model: in [15], the authors proved that the random spin configuration of a critical Ising model on $\mathbf{Z}^2$ has as a scaling limit a certain random generalized function (whose correlation functions are closely related to those of [18]), and it more or less follows that this gives rise to an operator and Hilbert space representation of the corresponding quantum field theory. This being said, as a probabilistic object, the scaling limit constructed in [15] is perhaps slightly poorly understood. Essentially no other characterisation for it is known besides being the scaling limit of the critical Ising model, or equivalently the unique random generalized function whose correlation functions are the scaling limit of the Ising ones. For example, if one wished to simulate it, to our knowledge, the easiest way is to simply simulate an Ising model on a domain with a fine mesh.

One of our goals is to show that if we change the model slightly, then one ends up with a random generalized function which can be constructed also in other ways – in particular, simulating it boils down to simulating a sequence of independent standard Gaussian random variables. The model we consider is the so-called XOR-Ising model (see e.g. [13, 83] and references therein for studies related to it), whose spin configuration is a pointwise product of two independent Ising spin configurations. Our main result concerning the XOR-Ising model is that for $d = 2$ and at the critical point, the spin configuration has a scaling limit which is the real part of an imaginary multiplicative chaos. The precise result is the following (for relevant definitions and notation concerning the XOR-Ising model, see Section 4.2 and for the Gaussian free field, see Example 2.6).

Theorem 1.5. *Let X be the zero boundary condition Gaussian free field on a simply connected bounded planar domain $U \subset \mathbf{R}^2$ and let $\mathcal{S}_\delta$ denote the spin field*[9] *of the XOR-Ising model on a lattice*

[8]Namely at a point of a phase transition where e.g. the correlation lengths of quantities of interest diverge.

[9]We find it convenient to define spin configurations as functions on faces of the lattice $\delta\mathbf{Z}^2$, or alternatively on the dual graph of $\delta\mathbf{Z}^2$, and by a spin field, we mean a function defined on U which is constant on these lattice faces and in each face, it agrees with the value of the spin configuration on that face.

approximation of U with δ-mesh and $+$ boundary conditions. Then for any $f \in C_c^\infty(U)$,

$$\delta^{-1/4} \int_U f(x)\mathcal{S}_\delta(x)dx \xrightarrow{d} \mathcal{C}^2 \int_U f(x) \left(\frac{2|\varphi'(x)|}{\operatorname{Im}\varphi(x)}\right)^{1/4} \cos\left(2^{-1/2}X(x)\right) dx$$

as $\delta \to 0$, where $\mathcal{C} = 2^{5/48}e^{\frac{3}{2}\zeta'(-1)}$, φ is any conformal bijection from U to the upper half plane, and $\cos\frac{1}{\sqrt{2}}X(x)$ denotes the real part of the random generalized function μ constructed in Theorem 1.1 from convolution approximations of the random generalized function X with $\beta = \frac{1}{\sqrt{2}}$, and the integral on the right hand side is formal notation meaning that we pair this random generalized function with the test function $f(x)(\frac{2|\varphi'(x)|}{\operatorname{Im}\varphi(x)})^{1/4}$.

We prove this theorem in Section 4.3. The proof follows rather easily from the strong results of [18], some rather rough estimates following arguments in [38], and the method of moments which is justified by Theorem 1.3. Rather interestingly, we note that our proof doesn't rely on anything converging to the GFF.

We emphasize here that the interpretation of Theorem 1.5 that one should have in mind is that if $\sigma_\delta(x)$ and $\widetilde{\sigma}_\delta(x)$ are the spin fields of two independent critical Ising realizations, then

$$\delta^{-1/4}\sigma_\delta(x)\widetilde{\sigma}_\delta(x) \overset{d}{\approx} \mathcal{C}^2 \left(\frac{2|\varphi'(x)|}{\operatorname{Im}\varphi(x)}\right)^{1/4} \cos\left(2^{-1/2}X(x)\right).$$

While studying the XOR-Ising model might seem like an artificial idea at first, it is in fact a model both physicists and mathematicians have studied and used to derive the scaling limit of the correlation functions of the critical Ising model and is referred to as bosonization of the Ising model. More precisely, in the physics literature, a connection between squared full plane Ising correlation functions and correlation functions of the cosine of the GFF were observed in [49] – for a review of later developments and more on the conformal field theory of the Ising model, see e.g. [27, Chapter 12]. This connection was given a rigorous basis in [31] where the author proved an exact identity between squares of Ising correlation functions and suitable correlation functions of the dimer model and then performed asymptotic analysis of these correlation functions. Intuitively, the connection to the free field comes from the fact that the relevant dimer correlation functions can be expressed in terms of the height function of the dimer model and it is known that this converges to the free field in the fine mesh limit.

Admittedly, for readers interested purely in the critical Ising model, our Theorem 1.5 is perhaps not much more than a curiosity showing that this notion of bosonization also makes rigorous probabilistic sense on the level of the scaling limit. This being said, we hope that from the perspective of better understanding scaling limits of critical models of statistical mechanics, Theorem 1.5 might be of some use, in that the cosine of the free field, interpreted in terms of imaginary multiplicative chaos, is a rather concrete object which might serve as a test case where proving some conjectured properties of scaling limits might be simpler than for other models – even the Ising model as everything is constructed in terms of Gaussian random variables. Although, we do concede that analytic and probabilistic results similar to those discussed in Section 1.2 have largely been proven for the critical Ising model; see [15, 16, 38]. Simulation on the other hand is certainly simpler for the scaling limit of the XOR-Ising model: see Figure 1.3 for a simulation of $\cos(2^{-1/2}X(x))$ in the unit square.

We now discuss an application of Theorem 1.5 to a model which is a perturbation of the critical XOR-Ising model. In addition to the connection between scaling limits of critical models of statistical physics and conformal field theory, physicists have argued that suitable perturbations of critical two-dimensional models of statistical mechanics should still have scaling limits described by quantum field theories which have an integrable structure despite loosing a conformal structure. For example, there exist fantastic conjectures concerning the scaling limit of the critical Ising model perturbed by a small magnetic field – see e.g. [84]. Another model where this type of structure is believed to exist is the so-called sine-Gordon model, which has been studied extensively in the physics literature (see

[10] More precisely, the eigenfunctions are of the form $\sin(k\pi x)\sin(\ell\pi y)$, $k, \ell \geq 1$, and we have used those for which $1 \leq k, \ell \leq 200$.

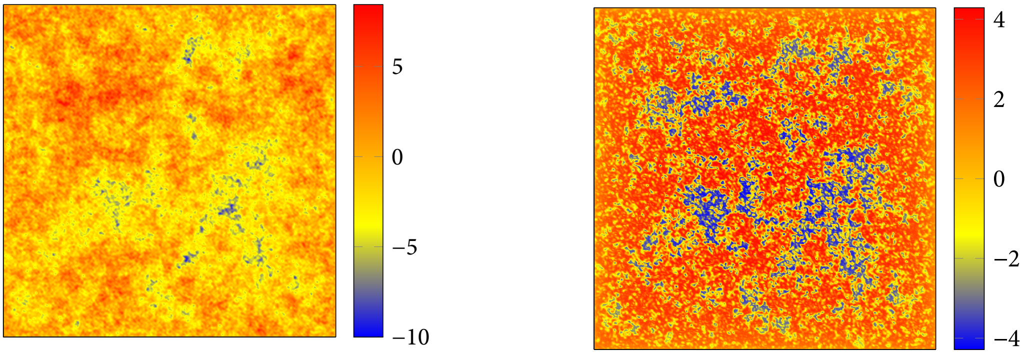


FIGURE 1. Left: A simulation of the Gaussian free field in the unit square with zero boundary conditions. The approximation is obtained by truncating the expansion in terms of Laplacian eigenfunctions at level 200^2 – see Example 2.6.[10] Right: A simulation of the cosine of the GFF obtained from the realization of the GFF in the left figure (with parameter $\beta = 1/\sqrt{2}$) – see Lemma 3.5.

e.g. [21, 62, 85]) and in the mathematical physics literature (see e.g. [37, 70, 28]). Formally, the probabilistic representation of the sine-Gordon model is a probability distribution on a suitable space of random generalized functions which is absolutely continuous with respect to the law of the full plane Gaussian free field X, with Radon–Nikodym derivative $\frac{1}{Z_{\beta,\mu}}e^{\mu\int\cos(\beta X(x))dx}$,[11] where $\mu, \beta \in \mathbf{R}$ and $Z_{\beta,\mu}$ is a normalizing constant. The conjectural integrable structure of this model is evident e.g. in [62], where it is conjectured that if $X_{\mathrm{sG}(\beta,\mu)}$ is distributed according to this law, then (formally – a rigorous statement would involve regularizing and taking a limit) for $0 < \beta < 2$ and $|\mathrm{Re}(\alpha)| < 2/\beta$.

$$\mathbb{E}\, e^{i\alpha X_{\mathrm{sG}(\beta,\mu)}(x)+\frac{\alpha^2}{2}\mathbb{E}[X_{\mathrm{sG}(\beta,\mu)}(x)^2]} = \left(\frac{\mu\pi\Gamma(1-\beta^2/4)}{2\Gamma(\beta^2/4)}\right)^{\frac{\alpha^2}{4-\beta^2}} e^{\int_0^\infty \left[\frac{\sinh^2\frac{\alpha\beta t}{2}}{2\sinh\frac{\beta^2 t}{4}\sinh t\cosh[1-\frac{\beta^2}{4}]t} - \frac{\alpha^2}{2}e^{-2t}\right]\frac{dt}{t}}.$$

Note that here truly one has $0 < \beta < 2$ instead of $\beta < \sqrt{2}$ as one would expect from e.g. Theorem 1.1. This is due to the fact that one can make sense of the sine-Gordon model also in this regime; the partition function $Z_{\beta,\mu}$ diverges, but correlation functions should be finite. We note that while slightly related to the convergence of imaginary chaos to white noise outside of the L^2-regime, this is a more delicate issue. At $\beta = 2$, there is a far more interesting transition for the sine-Gordon model than this L^2-boundary at $\beta = \sqrt{2}$ for imaginary chaos. This transition is known as the Kosterlitz–Thouless transition. We refer to [28, 70] and references therein for further information. We also point out that the condition $|\mathrm{Re}(\alpha)| < 2/\beta$ is simply the condition that the integral above converges.

While it currently seems that proving results of this flavor, or perhaps ones involving more complicated correlation functions are out of reach, we point out that this is surprisingly similar to quantities arising in Liouville field theory where significant progress has been made recently – compare e.g. with quantities appearing in the so-called DOZZ-formula in [56].

Our contribution to questions about near critical models of statistical mechanics and integrable quantum field theories is rather modest. First of all, we point out in Section 4.4, that in a finite domain and for suitable values of α, β, using results from Section 1.2, one can make sense of objects defined in the spirit of $e^{i\alpha X_{\mathrm{sG}(\mu,\beta)}(x)+\frac{\alpha^2}{2}\mathbb{E}[X_{\mathrm{sG}(\mu,\beta)}(x)^2]}$ – note that as the field $X_{\mathrm{sG}(\mu,\beta)}$ is non-Gaussian, this is an instance of non-Gaussian imaginary multiplicative chaos appearing naturally in a model of

[11]The precise definition of this is slightly delicate as the whole plane Gaussian free field is well defined only up to a random additive constant. Moreover, it is by no means clear that the "integral" here is convergent, or more precisely that the constant function one is a valid test function for the distribution, but as we are reviewing non-rigorous results due to physicists, we ignore this issue. A rigorous construction would involve first restricting to a bounded domain and then trying to take an infinite volume limit.

mathematical physics. After this, we observe that if one adds a (non-uniform) magnetic perturbation[12] to the critical planar XOR-Ising model – see Section 4.2 for proper definitions – then the spin field converges to the cosine of the sine-Gordon field in the scaling limit. More precisely, we have the following theorem (for proper definitions, see Section 4.2 and Section 4.4).

Theorem 1.6. *Let U be a bounded and simply connected domain and $f,\psi \in C_c^\infty(U)$. Let $\mathcal{S}_\delta$ be distributed according to the magnetically perturbed critical XOR-Ising model with magnetic field ψ on a lattice approximation of U with mesh δ and $+$ boundary conditions. Also write*

$$\widetilde{\psi}(x) = \mathcal{C}^2 \left(\frac{2|\varphi'(x)|}{\operatorname{Im}\varphi(x)}\right)^{1/4} \psi(x)$$

where $\mathcal{C}$ and φ are as in Theorem 1.5, and let $X_{\mathrm{sG}(\widetilde{\psi},1/\sqrt{2})}$ be distributed according to the sine-Gordon measure on U, written formally as

$$\frac{1}{Z_{\widetilde{\psi},\beta}} e^{\int_U \widetilde{\psi}(x)\cos[2^{-1/2}X(x)]dx}\mathbb{P}_{\mathrm{GFF}}(dX),$$

where $\mathbb{P}_{\mathrm{GFF}}(dX)$ denotes the law of the zero boundary condition Gaussian free field on U interpreted as a probability measure on say $H^{-\varepsilon}(\mathbf{R}^d)$.

Then as $\delta \to 0$, $\delta^{-1/4}\int_U \mathcal{S}_\delta(x) f(x)dx$ converges in law to a random variable written formally as

$$\mathcal{C}^2 \int_U f(x) \left(\frac{2|\varphi'(x)|}{\operatorname{Im}\varphi(x)}\right)^{1/4} \cos\left(2^{-1/2}X_{\mathrm{sG}(\widetilde{\psi},1/\sqrt{2})}(x)\right) dx.$$

We prove this theorem in Section 4.5. The proof follows rather easily from Theorem 1.5 and standard probabilistic arguments. The result is not very surprising given Theorem 1.5 and is certainly known in the physics literature, but we do point out that it seems difficult to prove a result of this flavor only from knowledge of the scaling limit of the critical correlation functions. Again our hope is that this type of result could be interesting as it provides a rather concrete case of a near critical model of statistical mechanics which has a scaling limit, conjectured to have an integrable structure and which is concrete enough that one might hope to be able to prove results that might be out of reach in more general models, or for example for the scaling limit of the magnetically perturbed Ising model.

This concludes the discussion of our main results concerning the Ising model and imaginary multiplicative chaos, so we turn to discussing imaginary multiplicative chaos in the setting of random matrix theory.

1.4. **Imaginary multiplicative chaos and random matrix theory.** In this section we review how log-correlated fields and real multiplicative chaos arise in random matrix theory and describe our result in the setting of imaginary multiplicative chaos.

In the last two decades, the connection between random matrix theory and log-correlated fields has been observed in various random matrix models – see e.g. [47, 74, 41]. More precisely, as the size of the matrix tends to infinity, the logarithm of the characteristic polynomial of a random matrix drawn from various distributions of unitary, hermitian, or normal matrices, is known to converge to a variant of the Gaussian free field after suitable recentering. As first utilized in [39, 40] on a heuristic level, one would then naturally expect that powers of the characteristic polynomial of such a random matrix should be related to the exponential of the Gaussian free field – multiplicative chaos. This type of results have since been proven for some models of random matrix theory – see e.g. [10, 58, 81] – though focusing on the case of e.g. real powers of the absolute value of the characteristic polynomial when the limiting object is a real multiplicative chaos measure.

In this article, we will consider large random unitary matrices drawn from the Haar measure on the unitary group, or in other words, we consider the so-called Circular Unitary Ensemble. Let us

[12] As can be seen from the definition in Section 4.2, adding a magnetic perturbation to the XOR-Ising model is different from taking pointwise products of two independent magnetically perturbed Ising models. Thus in this near critical case, one can't expect e.g. the correlation functions of the magnetically perturbed XOR-Ising model to be related to the original Ising model in any simple way.

write U_N for such a random $N \times N$ unitary matrix and consider two fields defined on the unit circle: define for $\theta \in [0, 2\pi]$

$$X_N(\theta) = \log |\det(I - e^{-i\theta}U_N)| \quad \text{and} \quad Y_N(\theta) = \lim_{r \to 1^-} \mathrm{Im}\mathrm{Tr} \log(I - re^{-i\theta}U_N),$$

where I denotes the $N \times N$ identity matrix, and in the definition of Y_N, what we mean by $\mathrm{Tr}\log(I - re^{-i\theta}U_N)$ is $\sum_{j=1}^{N} \log(1 - re^{i(\theta_j - \theta)})$, where $(e^{i\theta_j})_{j=1}^{N}$ are the eigenvalues of U_N, and the branch of the logarithm is the principal one – namely it is given by $\log(1-z) = -\sum_{k=1}^{\infty} \frac{1}{k} z^k$ for $|z| < 1$. Note that in this case, the limit defining Y_N exists almost surely e.g. in $L^2([0, 2\pi], d\theta)$. Thus the fields can be interpreted as the real and imaginary parts of the logarithm of the characteristic polynomial of U_N evaluated on the unit circle.

It was proven in [47] that as $N \to \infty$, X_N and Y_N converge in law to $2^{-1/2}$ times the 2d Gaussian free field restricted to the unit circle, namely a centered log-correlated Gaussian field X with covariance $\mathbb{E}\, X(\theta)X(\theta') = -\log|e^{i\theta} - e^{i\theta'}|$ – for details about this field, see Example 2.6. Moreover, this convergence was in the Sobolev space $H^{-\varepsilon}$ for arbitrary $\varepsilon > 0$ – this is essentially as nicely as a sequence of random generalized functions could converge. It was then proven in [81] that for $-\frac{1}{2} < \alpha < \sqrt{2}$ and $-\sqrt{2} < \beta < \sqrt{2}$, $\frac{e^{\alpha X_N(\theta)}}{\mathbb{E}\, e^{\alpha X_N(\theta)}} d\theta$ and $\frac{e^{\beta Y_N(\theta)}}{\mathbb{E}\, e^{\beta Y_N(\theta)}} d\theta$ converge in law to the multiplicative chaos measures formally written as $e^{\frac{\alpha}{\sqrt{2}} X(\theta)} d\theta$ and $e^{\frac{\beta}{\sqrt{2}} X(\theta)} d\theta$. In this article, we prove an analogue of this result for imaginary α and β. More precisely, the result is the following:

Theorem 1.7. *Let $X(\theta)$ be the log-correlated Gaussian field on $[0, 2\pi]$ with covariance $\mathbb{E}\, X(\theta)X(\theta') = -\log|e^{i\theta} - e^{i\theta'}|$ (see Example 2.6 for details), and $e^{i\beta X(\theta)}$ the associated imaginary multiplicative chaos distribution provided by Theorem 1.1. Then for any smooth and 2π-periodic $f : \mathbf{R} \to \mathbf{C}$*

$$\int_0^{2\pi} \frac{e^{i\beta X_N(\theta)}}{\mathbb{E}\, e^{i\beta X_N(\theta)}} f(\theta) d\theta \xrightarrow{d} \int_0^{2\pi} e^{i\frac{\beta}{\sqrt{2}} X(\theta)} f(\theta) d\theta$$

as $N \to \infty$, for $\beta \in (-\sqrt{2}, \sqrt{2})$. Moreover, as $N \to \infty$,

$$\int_0^{2\pi} \frac{e^{i\beta Y_N(\theta)}}{\mathbb{E}\, e^{i\beta Y_N(\theta)}} f(\theta) d\theta \xrightarrow{d} \int_0^{2\pi} e^{i\frac{\beta}{\sqrt{2}} X(\theta)} f(\theta) d\theta$$

for $\beta \in (-1, 1)$. In both statements, the integrals on the right hand side are formal notation meaning that the distribution $e^{i\frac{\beta}{\sqrt{2}} X(\theta)}$ is tested against f.

The proof of this Theorem 1.7, offered in Section 5, is in fact very similar to the one in [81] in the real case. Indeed, instead of the proof, what we hope readers will find interesting here is the discrepancy between the parameter values α and β for which convergence is obtained. We maintain that this is not a technical issue simply requiring better estimates, but truly that one does not have convergence for larger values of $|\beta|$ despite the fact that Y_N converges to a log-correlated field essentially as nicely as one might hope and that the corresponding multiplicative chaos exists. We suspect that this is due to the main part of Y_N behaving roughly like an integer valued function, see (5.1). We think these remarks should be viewed as a warning that one ought to take some care when hoping to prove that something converges to multiplicative chaos.

Finally we conclude this introduction with an outline of the remainder of the article.

1.5. **Outline of the article and acknowledgements.** In Section 2, we discuss some background material concerning log-correlated fields and their approximations and remind the reader about some basic definitions and properties of spaces of generalized functions. Then in Section 3, we prove our results from Section 1.2 concerning basic properties of imaginary multiplicative chaos. In Section 4, we prove our results on the Ising model while in Section 5, we prove our results concerning random matrix theory. In Appendix A, we record some basic moment bounds for imaginary chaos as well as a combinatorial counting argument we make use of in Section 3.

Acknowledgements: We wish to thank Antti Kupiainen for interesting discussions and references for the sine-Gordon model. We are also grateful to Julien Barral and Vincent Vargas for their comments on the manuscript. The first author was supported by the Doctoral Programme in Mathematics and Statistics at University of Helsinki. The first and second authors were supported by the Academy of Finland CoE 'Analysis and Dynamics', as well as the Academy of Finland Project 'Conformal methods in analysis and random geometry'. C.W. was supported by the Academy of Finland grant 308123.

2. Preliminaries: Introduction to log-correlated fields

In this section we give a precise definition of log-correlated Gaussian fields as random generalized functions, and discuss the type of approximations or regularizations of them that we shall use to construct our imaginary multiplicative chaos. More precisely, we realize log-correlated fields as random elements of suitable Sobolev spaces of generalized functions and define a class of approximations, containing e.g. convolution approximations, that are convenient for proving the existence of imaginary multiplicative chaos. The results of this section will probably look familiar to readers acquainted with basic facts about the Gaussian free field, as discussed e.g. in [30, Section 4] or [76], but unfortunately the definition and study of general log-correlated fields requires slightly heavier analysis than the GFF, especially in view of applications to imaginary chaos. In addition to discussing basic facts about log-correlated fields, we review in Section 2.2 the basic definitions and properties of spaces of generalized functions that we will need in this article. We have intended this section as an introduction to log-correlated fields for readers interested in generalities. Readers interested only in multiplicative chaos constructed from the Gaussian free field can skip the technical details of this section rather safely.

2.1. **Log-correlated fields.** Intuitively, we wish to construct a centered Gaussian process X on a domain $U \subset \mathbf{R}^d$ with covariance (kernel)

$$C_X(x,y) = \mathbb{E}\, X(x)X(y) = \log|x-y|^{-1} + g(x,y), \tag{2.1}$$

where we make the basic assumptions (used throughout the paper unless otherwise stated) that

$$\begin{cases} g \in L^1(U\times U) \cap C(U \times U), \quad g \text{ is bounded from above in } U\times U, \quad \text{and} \\ U \subset \mathbf{R}^d \quad \text{is a simpy connected and bounded domain.} \end{cases} \tag{2.2}$$

These assumptions cover some of the most common examples of log-correlated fields, but we expect that many of our results hold more generally too – in particular, one might hope to be able to relax the assumption of g being bounded from above to some degree. To avoid discussing in great detail generalized functions on domains with boundaries, we find it convenient to extend $C_X(x,y)$ to $\mathbf{R}^d \times \mathbf{R}^d$ by setting $C_X(x,y) = 0$ whenever $(x,y) \notin U \times U$. In addition, we also need to of course require that C_X is a covariance kernel, namely that it is symmetric and positive semi-definite: $C_X(x,y) = C_X(y,x)$ and

$$\int C_X(x,y) f(x)\overline{f(y)}\, dx\, dy \geq 0$$

for all $f \in C_c^\infty(\mathbf{R}^d)$. When a result needs more regularity to be assumed of g or U, this will be stated separately.

We note first that actually our conditions on C_X imply much stronger integrability of the covariance – we will make use of this to realize our process X as a random element in a suitable Sobolev space.

Lemma 2.1. *Assume that C_X is a covariance kernel satisfying* (2.1) *and* (2.2). *Then $C_X \in L^p(U\times U)$ for all $p < \infty$.*

Proof. Let $\psi_\varepsilon := \varepsilon^{-d}\psi(\cdot/\varepsilon)$, where $\psi \in C_c^\infty(\mathbf{R}^d)$ is a standard smooth, non-negative bump function with integral 1. We denote the mollified covariance by

$$C_{X_\varepsilon}(x,y) := \int_{\mathbf{R}^{2d}} \psi_\varepsilon(x-x')\psi_\varepsilon(y-y')C_X(x',y')dx'dy'.$$

From the definition of C_X, it easily follows that C_{X_ε} is a smooth honest covariance function (it will actually turn out to be the covariance of the mollified field $\psi_\varepsilon * X$, but we do not need this here). By smoothness, for any integer $p \geq 1$ also the power $(C_{X_\varepsilon})^p$ is a covariance, as is seen by considering products of independent copies of corresponding Gaussian fields. We apply the covariance condition on a smooth test function that is 1 on $U + B(0,1)$ and obtain for $\varepsilon \in (0,1)$ and any integer $p \geq 1$ the inequality $\int_{\mathbf{R}^{2d}} (C_{X_\varepsilon}(x,y))^p dxdy \geq 0$. By decomposing the covariance C_{X_ε} into its positive and negative part: $C_{X_\varepsilon} = (C_{X_\varepsilon})_+ - (C_{X_\varepsilon})_-$ and noting that $(C_{X_\varepsilon})_+ \leq [(C_X)_+]_\varepsilon$, it follows[13] that for any positive odd integer p

$$\begin{aligned}\int_{\mathbf{R}^{2d}} [(C_{X_\varepsilon})_-(x,y)]^p\, dxdy &\leq \int_{\mathbf{R}^{2d}} [(C_{X_\varepsilon})_+(x,y)]^p\, dxdy \ \leq\ \int_{\mathbf{R}^{2d}} [((C_X)_+)_\varepsilon(x,y)]^p\, dxdy \\ \leq\ \int_{\mathbf{R}^{2d}} [(C_X)_+(x,y)]^p\, dxdy\ =:\ c_p < \infty,\end{aligned}$$

where the last step follows by Minkowski's inequality and the assumption that g is bounded from above. Since $C_{X_\varepsilon} \to C_X$ almost everywhere as $\varepsilon \to 0^+$, we also see that almost everywhere, $(C_{X_\varepsilon})_- \to (C_X)_-$, and we may use Fatou's lemma to deduce that $\int_{\mathbf{R}^{2d}} ((C_X)_-(x,y))^p dxdy \leq c_p < \infty$. Again, since g is bounded from above, Minkowski's inequality implies now that $C_X \in L^p(U \times U)$ for arbitrary positive odd integers p and hence for all real $p \geq 1$. □

Remark 2.2. Using our assumption that $(C_X)_+(x,y) \leq c_0 + \log(1/|x-y|)$, the moment bound obtained in the proof may be used to deduce the stronger integrability $e^{(d-\varepsilon)|C_X|} \in L^1(U \times U)$ for every $\varepsilon > 0$. ■

The previous lemma verifies in particular that $(x,y) \mapsto C_X(x,y) \in L^2(\mathbf{R}^d \times \mathbf{R}^d)$, whence the operator $C_X \colon L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d)$ with the integral kernel $C_X(x,y)$ is Hilbert–Schmidt. In particular, it is symmetric and compact, so by the spectral theorem there exists a sequence $\lambda_1 \geq \lambda_2 \geq \cdots > 0$ of strictly positive eigenvalues and corresponding eigenfunctions φ_n, that together with those eigenfunctions that correspond to the eigenvalue 0 form an orthonormal basis for $L^2(\mathbf{R}^d)$. We will now formally define X via the (generalized) Karhunen–Loève expansion

$$X(x) := \sum_{n=1}^{\infty} A_n \sqrt{\lambda_n} \varphi_n(x), \qquad x \in \mathbf{R}^d, \tag{2.3}$$

where A_n are i.i.d. $N(0,1)$ random variables. Note that the functions φ_n are supported on U. Let us now show that this sum converges in a suitable Sobolev space of generalized functions – we refer the reader to Section 2.2 for the definition of the L^2-based standard Sobolev spaces $H^s(\mathbf{R}^d)$. While this result is well known for the GFF, and probably not very surprising to readers familiar with log-correlated fields, we choose to give a detailed proof of it here as it does not seem to appear in the literature.

Proposition 2.3. *The series on the right-hand side of* (2.3) *converges in $H^{-\varepsilon}(\mathbf{R}^d)$ for any $\varepsilon > 0$ to a $H^{-\varepsilon}(\mathbf{R}^d)$-valued Gaussian random variable with covariance kernel C_X satisfying* (2.1) *and* (2.2).

Proof. We start by showing that the series converges in $H^{-d/2-\varepsilon}(\mathbf{R}^d)$ for any $\varepsilon > 0$. Let $X_n(x) := \sum_{k=1}^n A_k \sqrt{\lambda_k} \varphi_k(x)$ denote the nth partial sum of (2.3). Then X_n form a $H^{-d/2-\varepsilon}(\mathbf{R}^d)$-valued martingale. As $H^{-d/2-\varepsilon}(\mathbf{R}^d)$ is a Hilbert space, it is enough to show that

$$\sup_{n \geq 1} \mathbb{E}\, \|X_n\|^2_{H^{-d/2-\varepsilon}} < \infty \tag{2.4}$$

in view of the almost sure convergence of Hilbert space valued L^2-bounded martingales (see e.g. [48, Theorem 3.61, Theorem 1.95]). For $f \in L^1(\mathbf{R}^d)$ we denote its Fourier transform by $\widehat{f}(\xi) := \int_{\mathbf{R}^d} f(x) e^{-2\pi i \xi \cdot x}\, dx$. Using elementary bounds along with orthogonality of the eigenfunctions, we may

[13] Here in the first step we use the fact that since $C_+ \cdot C_- = 0$ and p is odd, $0 \leq \int (C_+ - C_-)^p = \int C_+^p - \int C_-^p$, which is the first inequality. Bounding the norm of $(C_+)_\varepsilon$ with the norm of C_+ is justified e.g. by Young's convolution inequality.

compute

$$\begin{aligned}
\mathbb{E}\,\|X_n\|^2_{H^{-d/2-\varepsilon}} &= \int_{\mathbf{R}^d} \frac{\mathbb{E}\,|\widehat{X}_n(\xi)|^2}{(1+|\xi|^2)^{d/2+\varepsilon}}\,d\xi \le \int_{\mathbf{R}^d} \frac{\int_{U\times U} |\mathbb{E}\,X_n(x)X_n(y)|\,dx\,dy}{(1+|\xi|^2)^{d/2+\varepsilon}}\,d\xi \\
&\le C_\varepsilon \int_{U\times U} \Big|\sum_{k=1}^n \lambda_k\varphi_k(x)\varphi_k(y)\Big|\,dx\,dy \\
&\le C_\varepsilon |U| \left(\int_{\mathbf{R}^d\times\mathbf{R}^d} \Big|\sum_{k=1}^n \lambda_k\varphi_k(x)\varphi_k(y)\Big|^2\,dx\,dy\right)^{1/2} \\
&= C_\varepsilon |U| \sqrt{\sum_{k=1}^n \lambda_k^2} \;\le\; C_\varepsilon |U| \|C_X\|_{HS} < \infty
\end{aligned}$$

for some constant $C_\varepsilon > 0$ and $\|C_X\|_{HS}$ denoting the Hilbert–Schmidt norm of C_X. This proves (2.4).

Next we show that X actually takes values almost surely in $H^{-\varepsilon}(\mathbf{R}^d)$. We denote by $X_\delta := \psi_\delta * X$ a standard mollification of the field X (here ψ_δ is as in the proof of Lemma 2.1) whose covariance satisfies $C_{X_\delta} \in C_c^\infty(\mathbf{R}^{2d})$. Moreover, writing $a_\delta(x) := \int_{\mathbf{R}^d} C_{X_\delta}(u, u-x)\,du$ we have $a_\delta \in C_c^\infty(\mathbf{R}^d)$ and

$$\mathbb{E}\,|\widehat{X}_\delta(\xi)|^2 = \int_{\mathbf{R}^{2d}} C_{X_\delta}(x,y) e^{2\pi i\xi\cdot(y-x)}\,dx\,dy = \widehat{a}_\delta(\xi).$$

We compute for large enough p and small enough $\delta > 0$ that

$$\begin{aligned}
\mathbb{E}\,\|X_\delta\|^2_{H^{-\varepsilon}(\mathbf{R}^d)} &= \int_{\mathbf{R}^d} \frac{\mathbb{E}\,|\widehat{X_\delta}(\xi)|^2}{(1+|\xi|^2)^{\varepsilon}}\,d\xi \le \int_{\mathbf{R}^d} \frac{\mathbb{E}\,|\widehat{X_\delta}(\xi)|^2}{|\xi|^{2\varepsilon}}\,d\xi \\
&= \int_{\mathbf{R}^d} \frac{\widehat{a}_\delta(\xi)}{|\xi|^{2\varepsilon}}\,d\xi = c_\varepsilon \int_{\mathbf{R}^d} \frac{a_\delta(x)}{|x|^{d-2\varepsilon}}\,dx = c_\varepsilon \int_{U'^2} \frac{C_{X_\delta}(x,y)}{|x-y|^{d-2\varepsilon}}\,dx\,dy \\
&\le c'_{\varepsilon,p} \|C_{X_\delta}\|_{L^p(U'^2)} \le c'_{\varepsilon,p} \|C_X\|_{L^p(U^2)} < \infty,
\end{aligned}$$

where the last inequality is due to Lemma 2.1 and the second to last from Young's convolution inequality. Above $U' = U + B(0,1)$ and we used the fact that $(|\cdot|^{-2\varepsilon})^\wedge = c_\varepsilon|\cdot|^{-d+2\varepsilon}$. We then obtain

$$\mathbb{E}\,\|X\|^2_{H^{-\varepsilon}(\mathbf{R}^d)} = \mathbb{E}\,\lim_{\delta\to 0} \|X_\delta\|^2_{H^{-\varepsilon}(\mathbf{R}^d)} \le \liminf_{\delta\to 0} \mathbb{E}\,\|X_\delta\|^2_{H^{-\varepsilon}(\mathbf{R}^d)} \le c'_{\varepsilon,p}\|C_X\|_{L^p(U^2)} < \infty.$$

Finally, we lift the convergence $X_n \to X$ from $H^{-d/2-\varepsilon}(\mathbf{R}^d)$ to $H^{-\varepsilon}(\mathbf{R}^d)$. By the previous argument and by construction, the $H^{-\varepsilon}(\mathbf{R}^d)$-valued random variables X_n and $X - X_n$ are symmetric, independent, and their norms have finite variance. By considering $H^{-\varepsilon}(\mathbf{R}^d)$ as a real Hilbert space, the symmetry and independence yield for any $n \ge 1$

$$\begin{aligned}
\mathbb{E}\,\|X\|^2_{H^{-\varepsilon}(\mathbf{R}^d)} &= \mathbb{E}\,\|X - X_n\|^2_{H^{-\varepsilon}(\mathbf{R}^d)} + 2\mathbb{E}\,\langle X - X_n, X_n\rangle_{H^{-\varepsilon}(\mathbf{R}^d)} + \mathbb{E}\,\|X_n\|^2_{H^{-\varepsilon}(\mathbf{R}^d)} \\
&\ge\; \mathbb{E}\,\|X_n\|^2_{H^{-\varepsilon}(\mathbf{R}^d)}.
\end{aligned}$$

Thus (X_n) is a L^2-bounded $H^{-\varepsilon}(\mathbf{R}^d)$-valued martingale, which again yields the stated convergence. □

Remark 2.4. The existence of X as say a random tempered distribution could also be deduced by many other ways, e.g. it is a rather direct consequence of Bochner–Minlos' theorem (see e.g. [78, Theorem 2.3]). However, we wanted to avoid the more abstract framework and obtain directly the optimal Sobolev regularity. ■

To give the reader a sharper picture of what kind of objects log-correlated fields are, we discuss a bit further their smoothness properties. It is well-known and easy to show that the field X is almost surely not a Borel measure. However, it only barely fails being one, or even a function, since an arbitrarily small degree of smoothing makes X a continuous function. In order to make this precise, we recall that given $\delta \in \mathbf{R}$ there is a standard δ-lift operator I^δ that smoothes a given tempered distribution "by an amount of δ", see (2.20) below. Here is the exact statement concerning X being nearly a continuous function:

Lemma 2.5. *Let us assume that C_X is as in* (2.1) *and* (2.2)*. For any $\delta > 0$ there is an $\varepsilon > 0$ so that almost surely $I^\delta X \in C^\varepsilon(\mathbf{R}^d)$ – the space of ε-Hölder continuous functions. A fortiori, $X \in C^{-\varepsilon}(\mathbf{R}^d)$ for any $\varepsilon > 0$.*

Proof. We assume that $\delta \in (0,1)$. The covariance of $I^\delta X$ is given by $C_\delta := (G_\delta \otimes G_\delta) * C_X$, where G_δ is the so-called Bessel kernel, which is the integral kernel of the operator $(I-\Delta)^{-\delta/2}$ – see (2.20). Classical representations (see [4, (3,1)–(3,5), (4,1)]) of the Bessel-kernel G_δ imply that

$$G_\delta(x-y) = |x-y|^{\delta-d} H(|x-y|),$$

where H is an entire analytic function (as a side remark one may note that the main term in the resulting asymptotics has the same behaviour as the Riesz potential). Using this representation one can verify that given any $\delta > 0$, there is a $p_0(\delta) > 1$ and $\alpha > 0$ such that for $p \in (1, p_0(\delta))$ it holds that

$$\|(G_\delta \otimes G_\delta)(\cdot - x) - (G_\delta \otimes G_\delta)(\cdot)\|_{L^p(B\times B)} \lesssim |x|^\alpha$$

for any ball $B \subset \mathbf{R}^d$ and $x \in B \times B$. When this is combined with the fact that C_X has compact support and $C_X \in L^q(\mathbf{R}^{2d})$ for all $q < \infty$ by Lemma 2.1, one obtains by Hölder's inequality that the Gaussian field $I^\delta X$ has a Hölder-continuous covariance. In turn, this is well-known [5, Theorem 1.4.1] to imply that the realizations of $I^\delta X$ can be taken to be Hölder continuous.

The final statement then follows from basic properties of the operator I^δ, see the discussion around (2.20). ☐

In comparison, Proposition 2.3 states that X only barely fails being an L^2-function, while Lemma 2.5 states that X only barely fails being a Hölder continuous function, which is of course a stronger claim.

We now point out two examples of log-correlated Gaussian fields which will also play a role in our applications later on.

Example 2.6. Most common examples of log-correlated fields involve the two-dimensional Gaussian free field. While there are many related examples, we will consider the following two as they will be important in our applications to the Ising model and random matrices.

1. Let $U \subset \mathbf{R}^2$ be a bounded simply connected domain. Then the Gaussian free field on U with zero boundary conditions is the $\mathcal{D}'(\mathbf{R}^2)$-valued Gaussian random field with covariance

$$C_X(x,y) = G_U(x,y) = \log\left|\frac{1-\varphi(x)\overline{\varphi(y)}}{\varphi(x)-\varphi(y)}\right|, \tag{2.5}$$

   where G_U is the Green's function of the Laplacian in U with zero Dirichlet boundary conditions, and $\varphi : U \to \mathbb{D}$ is any conformal bijection. We could equivalently write $G_U(x,y) = \log\left|\frac{\psi(x)-\overline{\psi(y)}}{\psi(x)-\psi(y)}\right|$, where now $\psi : U \to \mathbb{H}^+$ is any conformal bijection from U to the upper half-pane. The generalized Karhunen–Loève expansion obtained in Proposition 2.3 lets us write

$$X(x) = \sum_{k=1}^\infty \frac{1}{\sqrt{\lambda_k}} A_k \varphi_k(x)$$

   with convergence in $H^{-\varepsilon}(\mathbf{R}^d)$ in the norm-topology. Here $(\lambda_k)_{k=1}^\infty$ are the eigenvalues of $-\Delta$, φ_k the associated eigenfunctions with unit L^2-norm (interpreted as zero outside of U), and $(A_k)_{k=1}^\infty$ i.i.d. standard Gaussians.

   The covariance given by the Green's function G_U satisfies condition (2.2) which may seen by applying the standard comparison $0 \le G_U(z,w) \le G_{U'}(z,w)$, where $U' \supset U$ is any larger simply connected domain and $z, w \in U$. The integrability and the needed upper bound are obtained via this inequality by picking a ball B such that $U \subset B$ and setting $U' = 2B$.
2. The trace of the whole plane Gaussian free field on the unit circle $\mathbb{T}$ is the $\mathcal{D}'(\mathbb{T})$-valued Gaussian random variable with covariance

$$C_X(z,w) = -\log|z-w|$$

with $|z| = |w| = 1$. Again X can be expressed in terms of a sum. Let $(W_k)_{k=1}^\infty$ be i.i.d. standard complex Gaussian random variables, i.e. $W_k = \frac{1}{\sqrt{2}}A_k + i\frac{1}{\sqrt{2}}B_k$ with $A_k, B_k \sim N(0,1)$ and i.i.d.. Then one has

$$X(z) = \sqrt{2}\mathrm{Re}\sum_{k=1}^\infty \frac{1}{\sqrt{k}} z^k W_k,$$

where the sum converges pointwise almost surely in $\mathcal{D}'(\mathbb{T})$ (again actually in $H^{-\varepsilon}(\mathbb{T})$ with respect to the norm topology for any $\varepsilon > 0$).

While the unit circle $\mathbb{T}$ is not an open subset of $\mathbf{R}^d$, we can say write $z = e^{ix}$ and take $x \in (-\pi, \pi)$ or something similar and see that the conditions (2.1) and (2.2) can be verified with various interpretations. ■

As X is a random generalized function and not an honest function, we need to define the exponential $e^{i\beta X}$ in terms of a renormalization procedure, where we smooth X into a function, exponentiate and then remove the smoothing. We will require our smoothing to have particular properties that are usually satisfied by most natural approximations of log-correlated fields (and are typical in the general theory of multiplicative chaos). We will call this type of an approximation a *standard approximation*:

Definition 2.7 (Standard approximation)**.** Let the covariance C_X be as in (2.1) and (2.2). We say that a sequence $(X_n)_{n\geq 1}$ of continuous jointly Gaussian centered fields on U is a standard approximation of X if it satisfies:

(i) One has

$$\lim_{(m,n)\to\infty} \mathbb{E}\, X_m(x)X_n(y) = C_X(x,y),$$

where convergence is in measure with respect to the Lebesgue measure on $U \times U$.

(ii) There exists a sequence $(c_n)_{n=1}^\infty$ such that $c_1 \geq c_2 \geq ... > 0$, $\lim_{n\to\infty} c_n = 0$, and for every compact $K \subset U$

$$\sup_{n\geq 1}\sup_{x,y\in K}\left|\mathbb{E}\, X_n(x)X_n(y) - \log\frac{1}{\max(c_n, |x-y|)}\right| < \infty.$$

(iii) We have

$$\sup_{n\geq 1}\sup_{x,y\in U}\left[\mathbb{E}\, X_n(x)X_n(y) - \log\frac{1}{|x-y|}\right] < \infty.$$

■

There can of course be various standard approximations. For example, one can check that for the GFF restricted to the unit circle from Example 2.6, one could take X_n to be the truncation of the sum at $k = n$ – see Example 2.9. Perhaps the most important class is provided by the usual mollifications of the field:

Lemma 2.8. *Let X be as in Proposition 2.3, and let $\eta \in C_c^\infty(\mathbf{R}^d)$ be non-negative, radially symmetric, with unit mass: $\int_{\mathbf{R}^d}\eta(x)dx = 1$, and with support $\mathrm{supp}(\eta) \subset B(0,1)$. For $x \in U$, $y \in \mathbf{R}^d$, and $\varepsilon > 0$ define $\eta_\varepsilon(y) = \varepsilon^{-d}\eta(y/\varepsilon)$ and set $X_\varepsilon(x) := X * \eta_\varepsilon(x) \times \mathbf{1}_U(x)$ for $x \in \mathbf{R}^d$.*[14]

Let $K \subset U$ be a compact set, $0 < \varepsilon < \delta$, and $x, y \in U$. We then have the estimates

$$\sup_{0<\varepsilon<\delta<1}\sup_{x,y\in K}\left|\mathbb{E}\, X_\varepsilon(x)X_\delta(y) - \log\frac{1}{\max(|x-y|,\delta)}\right| < \infty, \tag{2.6}$$

$$\lim_{\delta\to 0}\mathbb{E}\, X_\varepsilon(x)X_\delta(y) = C_X(x,y) \qquad \textit{for } x \neq y \textit{ fixed}, \tag{2.7}$$

$$\sup_{\varepsilon>0}\sup_{x,y\in U}\left[\mathbb{E}\, X_\varepsilon(x)X_\varepsilon(y) - \log\frac{1}{|x-y|}\right] < \infty, \tag{2.8}$$

and finally there exists a constant $C > 0$ depending only on K and η so that for $x, y \in K$

$$\mathbb{E}\,(X_\varepsilon(x) - X_\varepsilon(y))^2 \leq C|x-y|\varepsilon^{-1}. \tag{2.9}$$

[14] Recall that $X \in H^{-s}(\mathbf{R}^d) \subset \mathcal{S}'(\mathbf{R}^d)$ for any $s > 0$, so as $\eta_\varepsilon \in \mathcal{S}(\mathbf{R}^d)$, this convolution makes sense.

Especially, for any sequence $\delta_n \searrow 0$ the convolutions X_{δ_n}, $n \geq 1$, provide a standard approximation.

Proof. We begin with the proof of (2.6) and observe that by definition

$$\mathbb{E}\, X_\varepsilon(x) X_\delta(y) = \big((\eta_\varepsilon \otimes \eta_\delta) * C_X\big)(x,y)\mathbf{1}_{U\times U}(x,y). \tag{2.10}$$

Note that by our definition, C_X is extended to be zero outside $U \times U$, and C_X is integrable (actually belongs to all L^p-spaces by Lemma 2.1), so the convolution is well-defined in all of $\mathbf{R}^{2d}$. In turn, the factor $\mathbf{1}_{U\times U}$ verifies that the approximations are supported on U. Pick an open set V such that $K \subset V \subset \overline{V} \subset U$. Denote $a = a_K := \mathrm{dist}(K, \partial V) > 0$. Locally the function g is bounded uniformly from above and below on $\overline{V}$ by the assumed continuity, so its contribution to the convolution (2.10) is also uniformly bounded if $x, y \in K$ and $\varepsilon, \delta \leq a$. For other values of δ, ε the contribution of g is upper bounded by $\lesssim a^{-2d}$ from the integrability of g. Hence it remains to verify (2.6) just for the logarithmic term.

As the logarithmic term depends only on the difference $x - y$ we may write

$$\big((\eta_\varepsilon \otimes \eta_\delta) * \log(|\cdot - \cdot|^{-1})\big)(x,y) = \big((\eta_\varepsilon * \eta_\delta) * \log(|\cdot|^{-1})\big)(x-y). \tag{2.11}$$

Given any differentiable function $h : \mathbf{R}^d \to \mathbf{R}$ we have the easy estimate

$$\|\eta_\varepsilon * h - h\|_{L^\infty(B(x,r-\varepsilon))} \lesssim \varepsilon \|Dh\|_{L^\infty(B(x,r))} \tag{2.12}$$

for any $0 < \varepsilon < r$ and $x \in \mathbf{R}^d$. Let us denote $H := \eta_1 * \log(1/|\cdot|)$. As a smooth function H is uniformly bounded near the origin. Moreover, $|D\log(1/|x|)| \leq 1$ for $|x| \geq 1$, whence (2.12) yields that $|H(x) - \log(1/|x|)| \leq C$ for $|x| \geq 1$. These observation may be combined as follows:

$$\sup_{x\in\mathbf{R}^d} \big|H(x) - \log(1 \wedge |x|^{-1})\big| \leq C. \tag{2.13}$$

Using the smoothness of H and again the bound $|D\log(1/|x|)| \leq 1$ for $|x| \geq 1$, we see that $|DH|$ is uniformly bounded in $\mathbf{R}^d$, and hence (2.12) implies the inequality $\|\eta_\varepsilon * H - H\|_{L^\infty(\mathbf{R}^d)} < C$ uniformly in $\varepsilon \in (0,1)$. Putting things together we have shown that

$$\Big|\big((\eta_1 * \eta_\varepsilon) * \log(|\cdot|^{-1})(x) - \log(1 \wedge |x|^{-1}\big)\Big| \leq C \qquad \text{for all} \quad \varepsilon \in (0,1) \text{ and } x \in \mathbf{R}^d.$$

This is (2.6) for $1 = \delta > \varepsilon > 0$, and scaling yields the general case

$$\big|(\eta_\varepsilon * \eta_\delta) * \log(|\cdot|^{-1})(x) - \log\big(\frac{1}{\varepsilon \vee \delta \vee |x|}\big)\big| \leq C. \tag{2.14}$$

The convergence in (2.7) is immediate from standard properties of convolution and the continuity of C_X outside the diagonal. Next, (2.8) follows from (2.10), (2.11), (2.14) and the upper boundedness of g. Finally, for (2.9) we may clearly assume that $\varepsilon \leq a/2$ (where a depends on K as was defined in the beginning of the proof) and that g is continued as a uniformly bounded measurable function to the whole of $\mathbf{R}^d$ (the extension need not to be a covariance). For (2.9) it is enough to prove the derivative bounds $|D_x C_{X_\varepsilon}|, |D_y C_{X_\varepsilon}| \lesssim \varepsilon^{-1}$. Since $\int_{\mathbf{R}^d} |D\eta_\varepsilon| \lesssim \varepsilon^{-1}$, we obtain the stated bounds for the contribution of g to the derivative. In turn, for the contribution of the logarithm one assumes first that $\varepsilon = 1$. Then the uniform boundedness of the derivatives follow from (2.11) and the fact that $\|DH\|_\infty < \infty$, where H is as before. The case of general $\varepsilon \in (0,1)$ is again obtained by scaling.

Finally we note that conditions (i), (ii), and (iii) of a standard approximation follow from (2.6), (2.7), and (2.8). Thus we only need to check that (X_{δ_n}) are jointly Gaussian and continuous. We recall the simple argument for the convenience of a reader unfamiliar with such matters. By construction, all of the processes $(n,x) \mapsto X_{\delta_n}(x)$ live on the same probability space. Moreover, for any fixed $N \in \mathbf{Z}_+$, $x_1, ..., x_N \in U$, $n_1, ..., n_N \in \mathbf{Z}_+$, and $t_1, ..., t_N \in \mathbf{R}$,

$$\sum_{k=1}^N t_k X_{\delta_{n_k}}(x_k) = X\left(\sum_{k=1}^N t_k \eta_{\delta_{n_k}}(\cdot - x_k)\right)$$

and as we have e.g. $\sum_{k=1}^N t_k \eta_{\delta_{n_k}}(\cdot - x_k) \in H^\varepsilon(\mathbf{R}^d)$, this is a Gaussian random variable by definition, so indeed we have joint Gaussianity. Finally continuity follows by observing that $\eta_\delta(\cdot - x') \to \eta_\delta(\cdot - x)$ in $H^\varepsilon(\mathbf{R}^d)$ as $x' \to x$ and using the duality between $H^{-\varepsilon}$ and H^ε. $\square$

The proof of this result can be used to prove that other natural approximations are also standard approximations. As an example, we give the following one.

Example 2.9. Let $X_n(x) = \sqrt{2}\mathrm{Re}\sum_{k=1}^n \frac{1}{\sqrt{k}}e^{ikx}W_k$, where W_k are as in Example 2.6 part 2. Then the sequence $(X_n)_{n\geq 1}$ forms a standard approximation. Intuitively, this follows, since for the approximation

$$\widetilde{X}_n(x) := \sqrt{2}\mathrm{Re}\left(\sum_{k=1}^n \frac{\sqrt{n-k}}{\sqrt{nk}}e^{ikx}W_k\right)$$

we have $\mathbb{E}\,\widetilde{X}_n(x)\widetilde{X}_n(y) = \sum_{k=1}^n \frac{n-k}{nk}\cos(k(x-y))$. The last written sum is a convolution of the logarithmic kernel with a standard Fejér kernel, and the difference between the Fejér partial sum and Fourier partial sum is uniformly bounded by direct inspection. Finally, the Fejér partial sum of the logarithm is essentially a convolution approximation which behaves like the covariance of a standard approximation by the proof of Lemma 2.8. For a detailed argument, see e.g. the beginning of the proof of Lemma 6.5 in [51].

To conclude this preliminary section, we discuss briefly the spaces of generalized functions that we will discuss in this article.

2.2. **Classical function spaces.** Realizations of the imaginary chaos that we define in the next section are rather singular objects and one can't have convergence in any space of honest functions or even complex measures, so we must study convergence in suitable spaces of distributions. In fact this holds even true for log-correlated fields that were defined in the previous subsection, and therein we used the basic negative index Sobolev Hilbert spaces as a suitable tool. Here we recall for the convenience of readers less familiar with various spaces of generalized functions the definition of Sobolev spaces as well as of the other function spaces we use in the article.

For any smoothness index $s \in \mathbf{R}$ we define

$$H^s(\mathbf{R}^d) = \left\{\varphi \in \mathcal{S}'(\mathbf{R}^d) : \|\varphi\|_{H^s(\mathbf{R}^d)}^2 = \int_{\mathbf{R}^d}(1+|\xi|^2)^s\left|\widehat{\varphi}(\xi)\right|^2 d\xi < \infty\right\}, \tag{2.15}$$

where $\widehat{\varphi}$ stands for the Fourier transform of the tempered distribution φ – our convention for the Fourier transform is

$$\widehat{\varphi}(\xi) = \int_{\mathbf{R}^d} e^{-2\pi i\xi\cdot x}\varphi(x)dx$$

for any Schwartz function $\varphi \in \mathcal{S}(\mathbf{R}^d)$. Some basic facts about the spaces $H^s(\mathbf{R}^d)$ are e.g. that they are Hilbert spaces, for $s > 0$, $H^{-s}(\mathbf{R}^d)$ is the dual of $H^s(\mathbf{R}^d)$ with respect to the standard dual pairing, $H^s(\mathbf{R}^d)$ is a subspace of $C_0(\mathbf{R}^d)$ for $s > d/2$, i.e. there is a continuous embedding into the space of continuous functions vanishing at infinity, and for $s < -d/2$, compactly supported Borel measures (especially δ-masses) are elements of $H^s(\mathbf{R}^d)$.

A more extensive scale of measuring the simultaneous size and smoothness properties of functions is provided by Besov spaces on $\mathbf{R}^d$. In order to recall their definition, fix radial and non-negative Schwartz test functions $\phi_0, \phi_1 \in \mathcal{S}(\mathbf{R}^d)$, denote $\phi_k(x) := 2^{kd}\phi_1(2^k x)$ and assume that

$$\mathrm{supp}(\widehat{\phi_0}) \subset B(0,2), \qquad \mathrm{supp}(\widehat{\phi_1}) \subset B(0,4)\setminus B(0,1),$$

together with the partition of unity property $\sum_{k=0}^\infty \widehat{\phi}_k(\xi) = 1 \quad \text{for all} \quad \xi \in \mathbf{R}^d$. Assume that $1 \leq p,q \leq \infty$. A function (or Schwartz distribution) f on $\mathbf{R}^d$ belongs to the Besov space $B^s_{p,q}(\mathbf{R}^d)$ if

$$\|f\|_{B^s_{p,q}(\mathbf{R}^d)} := \left(\sum_{k=0}^\infty 2^{qks}\|\phi_k * f\|^q_{L^p(\mathbf{R}^d)}\right)^{1/q} < \infty, \tag{2.16}$$

where the interpretation for $q = \infty$ is $\|f\|_{B^s_{p,q}} := \sup_{k\geq 0} 2^{ks}\|\phi_k * f\|_{L^p(\mathbf{R}^d)}$. These spaces include many standard spaces. First of all, $B^s_{2,2}(\mathbf{R}^d) = W^{s,2}(\mathbf{R}^d) = H^s(\mathbf{R}^d)$. Moreover, if $s \in (0,1)$ we have

$B^s_{\infty,\infty}(\mathbf{R}^d) = C^s(\mathbf{R}^d)$ (with equivalent norms), where C^s is the well-known space of bounded Hölder continuous functions with the norm

$$\|f\|_{C^s(\mathbf{R}^d)} := \|f\|_{L^\infty(\mathbf{R}^d)} + \sup_{x,y\in\mathbf{R}^d} \frac{|f(x)-f(y)|}{|x-y|^s}.$$

Indeed, as is standard in harmonic analysis, one defines $C^s(\mathbf{R}^d) := B^s_{\infty,\infty}(\mathbf{R}^d)$ for arbitrary $s \in \mathbf{R}$.

Our motivation for proving in this paper basically optimal results for membership of the imaginary chaos in general Besov spaces comes from the fact that this yields considerably more knowledge on the smoothness and size of these objects than is obtained by just using the spaces $H^s(\mathbf{R}^d)$. Recall for example, that in the setting of log-correlated fields, our Proposition 2.3 said that the field X, if smoothed a little bit, becomes an L^2-function, which is far weaker than saying that it becomes continuous as was stated in Lemma 2.5. The latter result indeed measures smoothness using the Besov scale $B^s_{\infty,\infty}$, i.e. Hölder-spaces.

Another scale of function spaces is provided by the Triebel–Lizorkin spaces $F^s_{p,q}(\mathbf{R}^d)$, where we assume that $1 \le p, q < \infty$ and set

$$\|f\|_{F^s_{p,q}(\mathbf{R}^d)} := \Big\| \Big(\sum_{k=0}^\infty 2^{qks} |\phi_k * f|^q \Big)^{1/q} \Big\|_{L^p(\mathbf{R}^d)}.$$

This space contains as special cases e.g. the general Sobolev spaces $W^{k,p}(\mathbf{R}^d) = F^k_{p,2}(\mathbf{R}^d)$. However, we do not need to know more of them, since we will transfer our smoothness results from the Besov case to the Triebel–Lizorkin scale in view of the simple embeddings

$$B^{s+\delta}_{p,p}(\mathbf{R}^d) \subset F^s_{p,q}(\mathbf{R}^d) \subset B^{s-\delta}_{p,p}(\mathbf{R}^d) \tag{2.17}$$

which hold for any $\delta > 0$, all $1 \le p, q < \infty$ and $s \in \mathbf{R}$. This is easily shown from the very definitions of the spaces. For example, by Hölder's inequality we have for any sequence $(a_k)_{k\ge1}$ and $\delta > 0$ that $\|(a_k)_{k\ge1}\|_{\ell^{q'}} \lesssim \|(2^{k\delta}a_k)_{k\ge1}\|_{\ell^q}$ for any $q, q' \in [1,\infty]$. This shows that $F^s_{p,q}(\mathbf{R}^d) \subset F^{s-\delta}_{p,q'}(\mathbf{R}^d)$ for any q, q'. By choosing $q' = p$ and noting that $\|f\|_{F^s_{p,p}(\mathbf{R}^d)} = \|f\|_{B^s_{p,p}(\mathbf{R}^d)}$, we obtain the right hand inequality in (2.17), and the other one is proven in a similar way.

We need a couple of additional facts about Besov spaces. Fix $K \subset \mathbf{R}^d$ compact. Then for a distribution f in $\mathbf{R}^d$ with support contained in K we have also (now for the full range $1 \le p, q \le \infty$)

$$\|f\|_{B^s_{\infty,\infty}(\mathbf{R}^d)} \lesssim \|f\|_{B^{s'}_{p,p}(\mathbf{R}^d)} \quad \text{if} \quad s' \ge s + \frac{d}{p}. \tag{2.18}$$

and

$$\|f\|_{B^{s-\delta}_{1,1}(\mathbf{R}^d)} \lesssim \|f\|_{B^s_{p,q}(\mathbf{R}^d)} \lesssim \|f\|_{B^{s+\delta}_{\infty,\infty}(\mathbf{R}^d)}. \tag{2.19}$$

with the implied constants in (2.19) possibly depending on K. (2.18) is found in [80, Section 2.7.1.], and (2.19) follows by combining the reasoning from the end of the last paragraph with a standard expression for the Besov-norm using wavelets – see [65, Chapter 6]. One finally uses the simple fact that for functions f supported in a compact set K' we have by Hölder's inequality that $\|f\|_{L^{p_1}} \lesssim \|f\|_{L^{p_2}}$ for $1 \le p_1 \le p_2 \le \infty$.

For a subdomain $U \subset \mathbf{R}^d$ (naturally one may have $U = \mathbf{R}^d$) one says that a distribution $\lambda \in \mathcal{D}'(U)$ lies in the space $H^s_{loc}(U)$ if for all $\psi \in C^\infty_c(U)$ one has $\psi\lambda \in H^s(\mathbf{R}^d)$. In turn, one says that $\lambda \in H^s(U)$ assuming that there is $f \in H^s(\mathbf{R}^d)$ such that $\lambda = f_{|U}$ (then one defines $\|\lambda\|_{H^s(U)} := \inf\{\|f\|_{H^s(\mathbf{R}^d)} \mid \lambda = f_{|U}\}$). Similar conventions are used for other function spaces defined initially on $\mathbf{R}^d$.

One final general fact about the function spaces we will use is the standard δ-lift $I^\delta f$ ("smoothing by an amount δ") of a given $f \in \mathcal{S}'(\mathbf{R}^d)$, which for any fixed $\delta \in \mathbf{R}$ is defined by using the Fourier-transform as follows:

$$I^\delta f := \mathcal{F}^{-1}\Big((1+|\cdot|^2)^{-\delta/2}\widehat{f} \Big) = G_\delta * f, \tag{2.20}$$

where G_δ is the Bessel potential kernel. For any $\delta, s \in \mathbf{R}$ and $p, q \in [1, \infty]$ the map $I^\delta : B^s_{p,q}(\mathbf{R}^d) \to B^{s+\delta}_{p,q}(\mathbf{R}^d)$ is a continuous, linear and bijective isomorphism (see [80, Section 2.3.8]).

For an introduction to the basic properties of the L^2-Sobolev spaces, as well as for the Besov and Triebel spaces we refer in general to [44, Chapter 2], [80], [65].

This concludes our preliminary discussion about log-correlated fields and spaces of generalized functions. We will now move onto imaginary chaos.

3. Basic properties of imaginary multiplicative chaos

In this section, we prove our results stated in Section 1.2 concerning basic properties of imaginary multiplicative chaos as well as prove some auxiliary ones. We begin with Section 3.1 where we construct our imaginary multiplicative chaos and give some uniqueness results. In Section 3.2, we discuss stochastic properties of imaginary multiplicative chaos, namely we provide some general moment estimates, based on a generalisation of so-called Onsager type (electrostatic) inequalities (they will be discussed in Subsection 3.2 below) for general covariances with a logarithmic singularity on the diagonal. These are used to obtain uniqueness statements in terms of moments and tail estimates for the law of the imaginary chaos tested against a given test function. We then move on to proving basic estimates for the regularity of imaginary chaos in Section 3.3. Section 3.4 verifies that in the definition of "$e^{i\beta X}$" there is a lot of freedom in replacing $x \mapsto e^{ix}$ by another periodic function. Finally, in Section 3.5 we investigate what happens in the limit $\beta \nearrow \beta_{\text{crit}} = \sqrt{d}$. It is known from [57] that β_{crit} is the critical value for β beyond which the naive renormalization scheme of dividing $e^{i\beta X_n(x)}$ by $\mathbb{E}\, e^{i\beta X_n(x)}$ does not produce a non-trivial limiting object, and our Theorem 3.20 gives another manifestation of this fact.

3.1. **Construction of imaginary chaos.**

We begin by constructing imaginary multiplicative chaos and verifying some uniqueness properties, namely that the constructed object does not depend very much on the approximation used – see Theorem 1.1. Before starting, we recall that under a slightly more restrictive class of covariances C_X, the existence of the object follows already from results in [57], where complex multiplicative chaos was studied, but we offer a simple alternative proof here. We also mention that if one were to work for example in the class of tempered distributions, proving existence would be slightly simpler, but this would give very little insight into the regularity of these objects.

Let us start by proving existence. In our approach we are given a sequence of approximations $(X_n)_{n\geq 1}$ of the log-correlated field X on the domain U, which we use to define what we hope are approximations to our multiplicative chaos distribution:

$$\mu_n(x) := \exp\Big(\frac{\beta^2}{2}\mathbb{E}\,[X_n(x)^2] + i\beta X_n(x)\Big)\mathbf{1}_U(x).$$

We will first prove the convergence of μ_n in a suitable Sobolev space, assuming that X_n forms a standard approximation sequence as in Definition 2.7. As we will see in Section 3.3, the smoothness index we obtain here is not optimal, but we will return to finer regularity properties later. We also mention here that as follows from [57, Theorem 4.2] (under slightly more restrictive assumptions on g), one should not expect that μ_n has a limit for $\beta \geq \sqrt{d}$ unless it is multiplied by a suitable quantity tending to zero, in which case the limit should be proportional to white noise. As this is perhaps not as interesting a limiting object, we choose to focus on the regime $0 < \beta < \sqrt{d}$. The following proposition is the first ingredient of Theorem 1.1.

Proposition 3.1. *Let $(X_n)_{n\geq 1}$ be a standard approximation of a given log-correlated field X on a domain U (see Definition 2.7). When $0 < \beta < \sqrt{d}$, the functions μ_n converge in probability in $H^s(\mathbf{R}^d)$ for $s < -\frac{d}{2}$. The limit μ is a non-trivial random element of $H^s(\mathbf{R}^d)$, supported on $\overline{U}$.*

Proof. Assume first that $\varphi \in L^\infty(\mathbf{R}^d)$ is positive and let us write $C_{n,m}(x, y) = \mathbb{E}\, X_n(x)X_m(y)$, whence we have $C_{n,m}(x, y) = C_{m,n}(y, x)$. Then a short calculation shows that

$$\mathbb{E}\,|\mu_m(\varphi) - \mu_n(\varphi)|^2 \;=\; \int_U\int_U \varphi(x)\varphi(y)\Big(e^{\beta^2 C_{n,n}(x,y)} + e^{\beta^2 C_{m,m}(x,y)} - 2e^{\beta^2 C_{n,m}(x,y)}\Big)\,dx\,dy.$$

By (iii) of Definition 2.7, we have $e^{\beta^2 C_{n,n}(x,y)} = \mathcal{O}(|x-y|^{-\beta^2})$, where the implied constant is independent of x, y, n. Note that as $\beta^2 < d$, $|x-y|^{-\beta^2}$ is an integrable singularity (this is the role the $0 < \beta < \sqrt{d}$ condition plays). Thus by the dominated convergence theorem,

$$0 \leq \limsup_{(n,m)\to\infty} \mathbb{E}\,|\mu_m(\varphi) - \mu_n(\varphi)|^2 \;=\; \int_U \int_U \varphi(x)\varphi(y)(e^{\beta^2 C_X(x,y)} + e^{\beta^2 C_X(x,y)})\,dx\,dy$$
$$-\;\liminf_{(n,m)\to\infty} \int_U \int_U \varphi(x)\varphi(y)(e^{\beta^2 C_{n,m}(x,y)} + e^{\beta^2 C_{m,n}(x,y)})\,dx\,dy \;\leq\; 0,$$

where the last inequality follows by Fatou's lemma and property (i) of Definition 2.7. Thus we get

$$\lim_{(n,m)\to\infty} \mathbb{E}\,|\mu_m(\varphi) - \mu_n(\varphi)|^2 = 0,$$

implying that $\mu_n(\varphi)$ is a Cauchy sequence in $L^2(\mathbb{P})$. Moreover, by property (iii) of Definition 2.7, we have the simple upper bound

$$\mathbb{E}\,|\mu_m(\varphi) - \mu_n(\varphi)|^2 \leq \|\varphi\|_\infty^2 \int_U \int_U (e^{\beta^2 C_{n,n}(x,y)} + e^{\beta^2 C_{m,m}(x,y)})\,dx\,dy \leq C\|\varphi\|_\infty^2$$

for some constant $C > 0$. By splitting a complex valued φ into positive and negative real and imaginary parts we get the convergence in $L^2(\mathbb{P})$ of $\mu_n(\varphi)$ for all $\varphi \in L^\infty(\mathbf{R}^d)$[15], as well as the upper bound

$$\mathbb{E}\,|\mu_m(\varphi) - \mu_n(\varphi)|^2 \leq 16C\|\varphi\|_\infty^2. \tag{3.1}$$

We next compute

$$\begin{aligned}\mathbb{E}\,\|\mu_m - \mu_n\|_{H^s}^2 &= \int_{\mathbf{R}^d} (1+|\xi|^2)^s \mathbb{E}\,|\widehat{\mu_m}(\xi) - \widehat{\mu_n}(\xi)|^2\,d\xi \\ &= \int_{\mathbf{R}^d} (1+|\xi|^2)^s \mathbb{E}\,|\mu_m(e^{-2\pi i\xi\cdot}) - \mu_n(e^{-2\pi i\xi\cdot})|^2\,d\xi.\end{aligned}$$

Notice that if $s < -\frac{d}{2}$, then the estimate (3.1) and the dominated convergence theorem show us that as elements of H^s, the sequence is Cauchy in $L^2(\mathbb{P})$. Thus there exists a random element of H^s, say μ, living on the same probability space as our approximations, and satisfying $\mathbb{E}\,\|\mu\|_{H^s}^2 < \infty$ as well as $\lim_{n\to\infty} \mathbb{E}\,\|\mu_n - \mu\|_{H^s}^2 = 0$. In particular this implies convergence in probability in H^s of μ_n to μ.

Non-triviality of μ follows from L^2-convergence: one has e.g.

$$\mathbb{E}\,|\mu(\varphi)|^2 = \int_{U\times U} \varphi(x)\varphi(y)e^{\beta^2 g(x,y)}|x-y|^{-\beta^2}\,dxdy.$$

Finally, the claim of the support is evident since all the approximations μ_n are supported on $\overline{U}$ by definition. □

Having proven that limiting objects exist, the next natural step is to check that the limit μ does not depend on our approximating sequence μ_n in some sense. There are various statements of this flavor one could formulate; one example being that the law of the limit would be independent of the standard approximation. We return to such a question later with moments and now show with a simple argument that if there are two standard approximations living on the same probability space and are compatible in a certain way, then they converge in probability to the same random variable. The next proposition is the uniqueness portion of Theorem 1.1.

Proposition 3.2. *Suppose that X_n and $\widetilde{X}_n$ are two jointly Gaussian sequences of standard approximations of the same log-correlated field X and that*

$$\lim_{n\to\infty} \mathbb{E}\,X_n(x)\widetilde{X}_n(y) = C_X(x,y),$$

[15]Note that this result is essentially enough to ensure the existence of say a random tempered distribution μ_n converges to, but as stated before, it gives very little insight into the regularity of the object. Hence we work a bit harder to prove convergence in a Sobolev space, and later to extract the optimal regularity.

where the convergence takes place in measure on $U \times U$. Then the corresponding imaginary chaoses μ and $\widetilde{\mu}$ are equal almost surely.

Proof. It is enough to show that for all $f \in C_c^\infty(\mathbf{R}^d)$ we have

$$\lim_{n\to\infty} \mathbb{E}\,|\mu_n(f) - \widetilde{\mu}_n(f)|^2 = 0.$$

A straightforward computation shows that the expectation equals

$$\int_U \int_U f(x)\overline{f(y)}\big(e^{\beta^2 \mathbb{E}\, X_n(x)X_n(y)} + e^{\beta^2 \mathbb{E}\, \widetilde{X}_n(x)\widetilde{X}_n(y)} - e^{\beta^2 \mathbb{E}\, X_n(x)\widetilde{X}_n(y)} - e^{\beta^2 \mathbb{E}\, \widetilde{X}_n(x)X_n(y)}\big)\, dx\, dy.$$

Notice that since X_n and $\widetilde{X}_n$ are standard approximations, there exists a constant $c > 0$ such that on $U \times U$

$$e^{\beta^2 \mathbb{E}\, X_n(x)X_n(y)} + e^{\beta^2 \mathbb{E}\, \widetilde{X}_n(x)\widetilde{X}_n(y)} \leq \frac{c}{|x-y|^{\beta^2}}.$$

Thus by the reverse Fatou lemma we have

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{E}\,|\mu_n(f) - \widetilde{\mu}_n(f)|^2 \leq& \int_U \int_U f(x)f(y) \limsup_{n\to\infty} \big(e^{\beta^2 \mathbb{E}\, X_n(x)X_n(y)} + e^{\beta^2 \mathbb{E}\, \widetilde{X}_n(x)\widetilde{X}_n(y)} \\ &\qquad - e^{\beta^2 \mathbb{E}\, X_n(x)\widetilde{X}_n(y)} - e^{\beta^2 \mathbb{E}\, \widetilde{X}_n(x)X_n(y)}\big)\, dx\, dy \\ =&\ 0.\end{aligned}$$

□

By combining Propositions 3.1 and 3.2 we conclude the proof of Theorem 1.1.

Remark 3.3. Given a log correlated field X as in Proposition 2.3 and $\beta \in (0, \sqrt{d})$, when we speak of the imaginary chaos $\mu =$ "$\exp(i\beta X)$" we mean the chaos defined via Proposition 3.1 using convolution approximations. The definition is well-posed since convolution approximations yield a standard approximation according to Lemma 2.8, and the outcome does not depend on the approximation used as one may easily check that two different sequences of convolution approximations satisfy the conditions of Proposition 3.2. ■

In our application to the Ising model, what will turn out to be important is the real part of imaginary chaos. We now define this properly.

Definition 3.4. Given a log-correlated field X, satisfying our assumptions (2.1) and (2.2), and $\beta \in (0, \sqrt{d})$ the cosine of X (simply denoted by "$\cos(\beta X)$") is defined as the real part of the imaginary chaos, or in other words, for any test-function $\varphi \in C_c^\infty(\mathbf{R}^d)$ one has

$$\langle \cos(\beta X), \varphi\rangle := \lim_{n\to\infty} \int_U e^{\frac{1}{2}\beta^2 \mathbb{E}\,(X_n(x))^2} \cos(\beta X_n(x))\varphi(x) dx,$$

where the limit is in probability, and $(X_n)_{n\geq 1}$ is a sequence of convolution approximations of X. ■

The most important example of "$\cos(\beta X)$" is the one corresponding to a Gaussian free field (GFF) on a given simply connected planar domain $U \subset \mathbf{R}^2$, see the first part of Example 2.6. In Section 3.2 we shall characterise the laws of both "$\exp(i\beta X)$" and "$\cos(\beta X)$" via moments.

Before concluding this section about the existence and uniqueness of imaginary chaos, we mention that it is natural to ask whether the definition of the imaginary chaos could be done via the approximations given by the partial sums of the Karhunen–Loève expansion (2.3):

$$X_{KL,n}(x) := \sum_{k=1}^n A_k \sqrt{\lambda_k}\varphi_k(x). \tag{3.2}$$

The benefit of such a definition would be that it would allow using powerful probabilistic tools such as martingale theory and the Kolmogorov 0–1 law, which sometimes simplify proofs significantly. Unfortunately, checking even the uniform integrability condition (iii) in Definition 2.7 appears to be quite complicated in the case of the Karhunen–Loève approximations $X_{KL,n}(x)$, so we cannot refer to the above statements. However, under a mild further assumption, we will be able to settle the question by a more probabilistic argument.

Lemma 3.5. *Assume that $\beta \in (0, \sqrt{d})$ and that X is the GFF on a bounded simply connected subdomain of $\mathbf{C}$, or more generally, that X is a log-correlated field on a bounded domain in $\mathbf{R}^d$ with covariance satisfying our basic assumptions (2.1) and (2.2), and the additional size-condition $\sup_{x\in U} \|g(x,\cdot)\|_{L^2(U)} < \infty$. Denote $\nu_n(x) := \exp\big(\frac{1}{2}\beta^2\mathbb{E}\,[X_{KL,n}(x)^2] + i\beta X_{KL,n}(x)\big)$. As $n \to \infty$, ν_n converges to the imaginary chaos μ (see Remark 3.3). More specifically, given $\phi \in C_c^\infty(\mathbf{R}^d)$, we have as $N \to \infty$*

$$\langle \nu_n, \phi \rangle \to \langle \mu, \phi \rangle,$$

where the convergence is almost sure. Moreover, $\nu_m \to \mu$ almost surely in the Sobolev space $H^s(\mathbf{R}^d)$ for any $s < -d/2$.

Proof. We may assume that X is given by the Karhunen–Loève decomposition (2.3). Let us denote

$$Y_n := \langle \nu_n, \phi \rangle = \int_U \exp\big(\frac{1}{2}\beta^2\mathbb{E}\,[X_{KL,n}(x)^2] + i\beta X_{KL,n}(x)\big)\phi(x)dx,$$

whence Y_n is a martingale by construction. Here the integral is well-defined since by Cauchy–Schwarz, the condition $\sup_{x\in U} \|g(x,\cdot)\|_{L^2(U)} < \infty$ implies that each eigenfunction φ_k (corresponding to a non-zero eigenvalue) belongs to $L^\infty(U)$. In order to prove convergence of Y_n to something, as $n \to \infty$, the martingale structure implies that it is enough to verify that Y_n is L^2-bounded. Denote by X_{δ_k} a standard convolution approximation and note that since $X - X_{KL,n} \perp X_{KL,n}$ we may write $X_{\delta_k} = (X_{KL,n})_{\delta_k} + (X - X_{KL,n})_{\delta_k}$, where the summands are independent. This implies that

$$\mathbb{E}\Big(\exp\big(\frac{1}{2}\beta^2\mathbb{E}\,[X_{\delta_k}(x)^2] + i\beta X_{\delta_k}(x)\big)\big|\mathcal{F}_n\Big) = \exp\big(\frac{1}{2}\beta^2\mathbb{E}\,[(X_{KL,n})_{\delta_k}(x)^2] + i\beta(X_{KL,n})_{\delta_k}(x)\big), \tag{3.3}$$

where $\mathcal{F}_n$ is the σ-algebra generated by $\{A_1, \dots, A_n\}$, and A_i are the i.i.d. standard Gaussians from (3.2). By basic real analysis, as we are convolving L^1-functions with nice bump functions, there is a set $E \subset U$ of zero Lebesgue measure so that we have $(\varphi_j)_{\delta_k}(x) \to \varphi_j(x)$ for each j and $x \in U \setminus E$. Hence, if we denote

$$Y_{n,k} := \langle \nu_n, \phi \rangle = \int_U \exp\big(\frac{1}{2}\beta^2\mathbb{E}\,[(X_{KL,n})_{\delta_k}(x)^2] + i\beta(X_{KL,n})_{\delta_k}(x)\big)\phi(x)dx,$$

then we have $Y_{n,k} \to Y_n$ almost surely as $k \to \infty$. By dominated convergence and (3.3) it follows for every n that if we write μ_k for the approximation to μ given by X_{δ_k}, then

$$\mathbb{E}\,|Y_n|^2 \le \sup_k \mathbb{E}\,|Y_{n,k}|^2 = \sup_k \mathbb{E}\,\big|\mathbb{E}\left(\langle \mu_k, \phi\rangle\big|\mathcal{F}_n\right)\big|^2 \le \sup_k \mathbb{E}\,[|\langle \mu_k, \phi\rangle|^2] := C < \infty,$$

where the last inequality used again the uniform L^2-bound on approximations of μ coming from convolution approximations, which in turn followed from (2.8). Further, the above reasoning[16] also verifies that $Y_n = \mathbb{E}\left(\langle \mu, \phi\rangle|\mathcal{F}_n\right)$. Here both sides converge almost surely by the martingale property and L^2-boundedness, and the right hand side converges to $\langle \mu, \phi\rangle$ simply by the fact that $\langle \mu, \phi\rangle$ is measurable with respect to the σ-algebra $\sigma(\cup_{j=1}^\infty \mathcal{F}_j)$.

The stated convergence in the Sobolev space now follows since the above reasoning yields the uniform estimate $\mathbb{E}\,|Y_n|^2 \le c\|\phi\|_\infty^2$, which leads to ν_n being a L^2-bounded H^s-valued martingale. Finally, the GFF on a bounded planar domain $U \subset \mathbf{C}$ satisfies the extra size condition as we then have $0 \le C_X(z,w) \le c + \log(1/|z-w|)$ for any $z, w \in U$. ∎

This concludes our basic discussion about existence and uniqueness of imaginary chaos, and we move onto discussing probabilistic properties of imaginary chaos.

[16]More precisely: multiplying (3.3) by $\phi(x)$, integrating over U, and letting $k \to \infty$, one sees that the left hand side of (3.3) becomes $\mathbb{E}\left(\langle \mu, \phi\rangle|\mathcal{F}_n\right)$ – this used the fact that $\mu_k \to \mu$ in L^2. On the other hand, before taking the $k \to \infty$ limit, the right hand side equals $Y_{n,k}$ and we saw that this tends to Y_n as $k \to \infty$.

3.2. **Moment and tail bounds.** In this section we will prove moment and tail bounds for imaginary chaos, namely Theorem 1.3 and Theorem 1.4. The situation is quite different from real chaos (or complex chaos in general), since, as we will see in this section, for μ from Theorem 1.1, the moments $\mathbb{E}\,|\mu(f)|^{2N}$ are finite for all $N \geq 1$ and all $f \in C_c^\infty(U)$. Moreover, it will turn out that (under minor smoothness assumptions on g from (2.1)) these moments grow slowly enough for one to be able to characterize the law of $\mu(f)$ in terms of its moments. This makes proving that something converges to imaginary chaos rather straightforward since it is then a question about controlling moments – indeed, this is what we will show for the XOR-Ising model.

Before going into details about the moments, let us point out that a (formal) straightforward Gaussian computation yields the formula

$$\mathbb{E}\,|\mu(f)|^{2N} \text{ “}=\text{” } \int_{U^{2N}} \frac{\prod_{1\leq i<j\leq N} e^{-\beta^2 C_X(x_i,x_j)} \prod_{1\leq i<j\leq N} e^{-\beta^2 C_X(y_i,y_j)}}{\prod_{1\leq i,j\leq N} e^{-\beta^2 C_X(x_i,y_j)}} \prod_{i=1}^N f(x_i)\overline{f(y_i)}dx_i dy_i, \tag{3.4}$$

where we have written “ = ” to indicate that we have not justified this identity beyond $N = 1$, or that one would have convergence of say μ_δ to μ in all L^p-spaces. Nevertheless, let us not worry about rigor for a moment. The archetypical case of (3.4) would be $C_X(x, y) = \log \frac{1}{|x-y|}$ and $f \equiv 1$ (or more precisely, $f \in C_c^\infty(\mathbf{R}^d)$ and $f|_U = 1$), in which case (3.4) becomes the following interesting integral:

$$\int_{U^{2N}} \frac{\prod_{1\leq i<j\leq N} |x_i - x_j|^{\beta^2} \prod_{1\leq i<j\leq N} |y_i - y_j|^{\beta^2}}{\prod_{1\leq i,j\leq N} |x_i - y_j|^{\beta^2}} \, dx_1 \dots dx_N dy_1 \dots dy_N \tag{3.5}$$

The finiteness of (3.4) for all $\beta \in (-\sqrt{d}, \sqrt{d})$ is not completely trivial, although it is well-known to experts and can be proven e.g. by using the techniques in [57, Appendix A]. Rather precise lower and upper bounds for (3.5) are known for $d = 2$, see e.g. [45, 61]. As we will see later on, these bounds imply in particular that the law of $\mu(f)$ is determined by its moments. Our goal in this section is to prove similar bounds in all dimensions and for more general covariance kernels. This is also crucial for us in Section 4, where we deal with the convergence of the XOR-Ising model. Note that in this case, the relevant field is the zero boundary condition GFF from Example 2.6 and moment bounds on the corresponding imaginary chaos do not follow directly e.g. from [45, 61].

In [45] estimates for moments in the case of the purely logarithmic kernel are obtained via first establishing a 2-dimensional version of a famous inequality called Onsager's lemma [71] (also sometimes called the electrostatic inequality). The original 3-d version of Onsager's inequality (where one has the $|x|^{-1}$-kernel instead of our logarithmic kernel) has been used e.g. in the modern theory of stability of matter [34, 36], and we refer to [36] or [79] for a mathematical proof of the inequality. These proofs do not apply as such for our general logarithmic covariance kernels, especially in the case of $d \neq 2$, but we will shortly discuss in more detail how this can be overcome and explain the various versions of the generalised inequality we shall need.

In any case, after a suitable version of Onsager is at our hand, we may then finish the proof of the desired moment bounds by implementing the combinatorial part of the argument in [45] as stated in Lemma 3.10 below. We include a proof of the lemma in the appendix for the reader's convenience as the proof in [45] is for $d = 2$ and there are cosmetic differences for $d \neq 2$. Moreover, we also note that the approach of [45] for lower bounds of the moments generalizes to some extent, and we record consequences for the tail of the imaginary chaos. Finally, it is to be noted that very precise estimates for the moments in the case of $d = 2$ and the purely logarithmic kernel were obtained recently in [61], with applications to the tails of the corresponding imaginary chaos.

Let us then discuss our versions of Onsager's lemma, of which there are four in total. Our first version (see Proposition 3.6 (i) below) takes care of general 2-dimensional covariances for which $g \in C^2(U \times U)$. This generalizes the one in [45], which considers just the purely logarithmic kernel. To achieve this generalization, we need to replace the complex analytic proof of [45] by a more probabilistic one. The effect of the term g in the covariance is dealt with by a rather direct error analysis. Surprisingly enough, this proof or the other known ones appear not to work for dimensions $d \neq 2$, and for that purpose we require a more complicated approach based on a general decomposition principle of logarithmic covariances – indeed, our second version of Onsager's inequality is Theorem

3.8 below, and its proof will be published elsewhere as it relies on the above decomposition principle whose proof we feel does not belong in this article. The above versions of Onsager are local in the sense that one considers points lying in a fixed subset of U. In contrast, our third version (Proposition 3.9 below) is a global result in the case of the GFF on a bounded domain. Finally, our fourth version (Proposition 3.6 (ii) below) is an auxiliary result that does not require further regularity from g, but comes at the cost of having error of order $O(N^2)$ instead of $O(N)$. Hence it is not an 'honest Onsager inequality' from our point of view. In fact, quadratic error in N is too large to prove that the moments determine the distribution, but we may use this version of the inequality to verify that $\mathbb{E}\,|\mu_\varepsilon(f)|^{2N}$ converges to (3.4) as $\varepsilon \to 0$, validating our formal computations and verifying that all moments are finite.

We start with the first and fourth version of our Onsager inequalities.

Proposition 3.6. *Let K be a compact subset of U, $N \geq 1$, $q_1, \dots, q_N \in \{-1, 1\}$, and $x_1, \dots, x_N \in K$. Assume that the covariance of X is as in* (2.1) *and that g satisfies the assumptions* (2.2). *We then have the following two Onsager-type inequalities*:

(i) *Let $d = 2$ and assume that in addition to* (2.2) *we have $g \in C^2(U \times U)$. Then*

$$-\sum_{1 \leq j < k \leq N} q_j q_k \mathbb{E}\, X(x_j) X(x_k) \leq \frac{1}{2} \sum_{j=1}^{N} \log \frac{1}{\frac{1}{2} \min_{k \neq j} |x_j - x_k|} + CN,$$

for some constant $C > 0$ depending only on g and K.

(ii) *Let $d \geq 1$ be arbitrary. For convolution approximations X_ε* (*as in Lemma 2.8*) *of X we have*

$$-\sum_{1 \leq j < k \leq N} q_j q_k \mathbb{E}\, X_\varepsilon(x_j) X_\varepsilon(x_k) \leq \frac{1}{2} \sum_{j=1}^{N} \log \frac{1}{\frac{1}{2} \min_{k \neq j} |x_j - x_k|} + CN^2$$

for some constant $C > 0$ that is independent of $\varepsilon > 0$, and depends only on g and K. Note that no extra assumptions beyond (2.2) *on g are required in this case.*

Proof. Let $r_j = \frac{1}{2}\big(\min_{k \neq j} |x_j - x_k| \wedge \operatorname{dist}(K, \partial U)\big)$ and set (see here Remark 3.7)

$$Z_j = \frac{1}{2\pi} \int_0^{2\pi} X(x_j + r_j e^{i\theta})\, d\theta.$$

We have

$$\begin{aligned} \mathbb{E}\, Z_j^2 &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \Big(\log \frac{1}{|r_j e^{i\theta} - r_j e^{i\varphi}|} + g(x_j + r_j e^{i\theta}, x_j + r_j e^{i\varphi}) \Big)\, d\theta\, d\varphi \\ &= \log \frac{1}{r_j} + \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} g(x_j + r_j e^{i\theta}, x_j + r_j e^{i\varphi})\, d\theta\, d\varphi \end{aligned} \tag{3.6}$$

by harmonicity of $\log(|\cdot|^{-1})$. Moreover, for $j \neq k$ we obtain, again using harmonicity of the log,

$$\begin{aligned} \mathbb{E}\, Z_j Z_k &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \Big(\log \frac{1}{|x_j + r_j e^{i\theta} - x_k - r_k e^{i\varphi}|} + g(x_j + r_j e^{i\theta}, x_k + r_k e^{i\varphi}) \Big)\, d\theta\, d\varphi \\ &= \log \frac{1}{|x_j - x_k|} + \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} g(x_j + r_j e^{i\theta}, x_k + r_k e^{i\varphi})\, d\theta\, d\varphi. \end{aligned}$$

Letting $c_{j,k} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} g(x_j + r_j e^{i\theta}, x_k + r_k e^{i\varphi})\, d\theta\, d\varphi$ this means that

$$\mathbb{E}\, Z_j^2 = \log \frac{1}{r_j} + c_{j,j} \quad \text{and} \quad \mathbb{E}\, Z_j Z_k = \log \frac{1}{|x_j - x_k|} + c_{j,k}.$$

A simple computation (where we allow also $j = k$) yields that

$$\begin{aligned}
c_{j,k} &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} g(x_j + r_j e^{i\theta}, x_k + r_k e^{i\varphi})\, d\theta\, d\varphi \\
&= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \Big(g(x_j, x_k) + \begin{pmatrix} r_j e^{i\theta} \\ r_k e^{i\varphi}\end{pmatrix} \nabla g(x_j, x_k) + \xi(r_j e^{i\theta}, r_k e^{i\varphi})\Big)\, d\theta\, d\varphi \\
&= g(x_j, x_k) + \int_0^{2\pi}\int_0^{2\pi} \xi(r_j e^{i\theta}, r_k e^{i\varphi})\, d\theta\, d\varphi, \;=:\; g(x_j, x_k) + d_{j,k},
\end{aligned}$$

where ξ is the remainder in the Taylor expansion of g at the point (x_j, x_k), and the error $d_{j,k}$ is of the order

$$|d_{j,k}| \lesssim \max(r_j^2, r_k^2).$$

Since Z_j are jointly Gaussian, their covariance is positive definite, and in particular

$$\begin{aligned}
0 &\le \sum_{1\le j,k\le N} q_j q_k \mathbb{E}\, Z_j Z_k = \sum_{j=1}^N \mathbb{E}\, Z_j^2 + \sum_{j\neq k} q_j q_k \mathbb{E}\, Z_j Z_k \\
&= \sum_{j=1}^N \log \frac{1}{\frac{1}{2}(\min_{k\neq j}|x_k - x_j| \wedge \operatorname{dist}(K, \partial U))} + 2\sum_{1\le j<k\le N} q_j q_k \mathbb{E}\, X(x_j)X(x_k) \\
&\quad + \sum_{j=1}^N d_{j,j} + 2\sum_{1\le j<k\le N} q_j q_k d_{j,k}.
\end{aligned}$$

A key observation for the proof is that by the disjointness of the circles and since $d = 2$ we have the area estimate

$$|\sum_{j=1}^N d_{j,j}| \lesssim \sum_{j=1}^N r_j^2 \lesssim |U|. \tag{3.7}$$

In turn,

$$\log \frac{1}{\frac{1}{2}\big(\min_{k\neq j}|x_k - x_j| \wedge \operatorname{dist}(K,\partial U)\big)} \le \log \frac{1}{\frac{1}{2}\min_{k\neq j}|x_k - x_j|} + \max\big(\log\frac{1}{\frac{1}{2}\operatorname{dist}(K,\partial U)}, 0\big).$$

Moreover, (3.7) implies that

$$|\sum_{1\le j<k\le N} q_j q_k d_{j,k}| \le \sum_{1\le j<k\le N} c\max(r_j^2, r_k^2) \le 2Nc|U|$$

for some constant $c > 0$ that depends on g. By putting all the observations together, part (i) of the claim follows.

In order to prove the second inequality, we again employ auxiliary random variables Z_j. Letting the radii r_j be as before we set this time

$$Z_j := X_{\max(\varepsilon, r_j)}(x_j).$$

By Lemma 2.8 we have

$$\mathbb{E}\, Z_j^2 = \log\frac{1}{\max(\varepsilon, r_j)} + O(1)$$

and

$$\mathbb{E}\, Z_j Z_k = \log\frac{1}{\max(\varepsilon, |x_j - x_k|)} + O(1) = \mathbb{E}\, X_\varepsilon(x_j)X_\varepsilon(x_k) + O(1).$$

Hence

$$0 \le \sum_{1\le j,k\le N} q_j q_k \mathbb{E}\, Z_j Z_k = \sum_{j=1}^N \mathbb{E}\, Z_j^2 + \sum_{j\neq k} q_j q_k \mathbb{E}\, Z_j Z_k$$
$$\le \sum_{j=1}^N \log \frac{1}{\log(\frac{1}{2}(\min_{k\neq j}|x_k - x_j|))} + 2\sum_{1\le j<k\le N} q_j q_k \mathbb{E}\, X_\varepsilon(x_j) X_\varepsilon(x_k) + CN^2.$$

□

Remark 3.7. Note that the definition of the variables Z_j in the above proof is somewhat formal; we have only defined X as an element of $H^{-\varepsilon}(\mathbf{R}^2)$, so it so it would seem that integrating X over a circle can not be interpreted as X acting on a valid test function. Nevertheless, the probabilistic objects we use are simply a device to obtain covariance inequalities. To make things precise, one might want to rephrase the definition of Z_j as $Z_j := X(\rho_{\varepsilon,x_j})$, where $\rho_{\varepsilon,x_j} \in C_c^\infty(\mathbf{R}^2)$ is a convolution approximation of uniform probability measure on a circle of radius r_j around x_j. Then later in the obtained covariance inequalities, one simply lets $\varepsilon \to 0$ and gets the desired statements. However, we feel that this level of precision could obscure the idea of the proof and hope that the reader will be forgiving us for the slight inaccuracy in the exposition.

Let us next state the third version of Onsager's lemma, which is even more local in nature than Proposition 3.6 but works in arbitrary dimensions. For a definition of the space H^s_{loc}, we refer the reader to Section 2.2.

Theorem 3.8. *Assume that X is a log-correlated field on the domain $U \subset \mathbf{R}^d$ with $0 \in U$ and assume that $g \in H^{d+\varepsilon}_{loc}(U \times U)$ for some $\varepsilon > 0$. Then there is a neighbourhood $B_\delta(0) \subset U$ of the origin so that X satisfies the following electrostatic inequality in $B_\delta(0)$:*
for any $N \ge 1$, $q_1, \dots, q_N \in \{-1, 1\}$ and $x_1, \dots, x_N \in B_\delta(0)$ it holds that

$$-\sum_{1\le j<k\le N} q_j q_k \mathbb{E}\, X(x_j) X(x_k) \le \frac{1}{2}\sum_{j=1}^N \log \frac{1}{\frac{1}{2}\min_{k\neq j}|x_j - x_k|} + CN, \tag{3.8}$$

where C is independent of the points x_j or N, but may depend on the neighbourhood $B_\delta(0)$.

Proof. This is Theorem 6.1 in [52]. □

One should observe that in the above result, in case $d = 2$ (disregarding the more local nature that does not affect our moment estimates) the condition on g is certainly satisfied if $g \in C^{2+\varepsilon}$. On the other hand, in a certain sense the class $H^{2+\varepsilon}_{loc}(\mathbf{R}^2 \times \mathbf{R}^2)$ is much larger than $C^2(\mathbf{R}^2 \times \mathbf{R}^2)$, e.g. it allows for local behaviour of type $|x - x_0|^\delta$, $\delta > 0$, so the conditions are not comparable but extend each other.

All the above results are local in nature. In order to obtain full grip of the moments, or optimal understanding of the imaginary chaos on a two-dimensional bounded domain as a random element in $\mathcal{S}'(\mathbf{R}^d)$, it is desirable to have a global version which is valid for all $x_1, \dots, x_N \in U$. This can be achieved as a consequence of the previous results if g continues with suitable smoothness in a neighbourhood of the closure $\overline{U}$ (by Theorem 3.8 the extension needs not to be even a covariance). We next show that one can also obtain a global Onsager inequality in the case of the GFF on a bounded simply connected domain $U \subset \mathbf{R}^2 = \mathbf{C}$. For that end let us recall that the density of the hyperbolic metric of U at a point $z \in U$ is given by

$$|d_H z| := \frac{2|\psi'(z)|}{1 - |\psi(z)|^2}|dz|,$$

where $\psi : U \to \mathbb{D}$ is any conformal map. The hyperbolic distance between two points in U is obtained by minimizing the integral $\int_\gamma |d_H z|$ over all rectifiable curves in U joining the given points. In a simply connected domain the classical Koebe estimate ([42, Theorem 4.3] – we refer overall to [42] on basic facts on hyperbolic metric) says that

$$\frac{1}{2}(d(z, \partial U))^{-1}|dz| \le |d_H z| \le 2(d(z, \partial U))^{-1}|dz|.$$

In particular, the hyperbolic distance dominates a multiple of the standard metric. The hyperbolic metric is conformally invariant, whence one easily computes that in the unit disc the hyperbolic distance of points $w, z \in \mathbb{D}$ equals

$$d_H(w,z) = \log\Big(\frac{1+\rho(w,z)}{1-\rho(w,z)}\Big), \qquad \text{with} \quad \rho(w,z) := \Big|\frac{z-w}{1-\overline{z}w}\Big|,$$

where $\rho(w,z)$ is called the pseudo hyperbolic metric between z and w. Also ρ is an honest metric. Given $z_0 \in U$ and $r > 0$ we denote by $B_\rho(z_0, r) \subset U$ the pseudo-hyperbolic ball of radius r. We then have $B_\rho(z_0, r) = B_H(z_0, r')$, and this is the image of the ordinary ball $B(0, R) \subset \mathbb{D}$ under any conformal map $\psi^{-1} : \mathbb{D} \to U$ such that $\psi(z_0) = 0$. Here $R = r$ and r' is given by $r' = \log((1 + r)/(1 - r))$.

Proposition 3.9. *Assume that $U \subset \mathbf{R}^2$ is simply connected and bounded and that X is the zero boundary condition GFF on U. Let $N \geq 1$, $q_1, \ldots, q_N \in \{-1, 1\}$, and $x_1, \ldots, x_N \in U$ be arbitrary. Then*

$$-\sum_{1\leq j<k\leq N} q_j q_k \mathbb{E}\, X(x_j)X(x_k) \leq \frac{1}{2}\sum_{j=1}^{N} \log\left(\frac{1}{\frac{1}{2}\min_{k\neq j}|x_j - x_k|}\right) + CN$$

for some constant $C > 0$ depending only on the domain U.

Proof. We assume first that $U = \mathbb{D}$. Let $r_j = \frac{1}{2}\inf_{k\neq j} d_\rho(x_j, x_k)$ be half the pseudo hyperbolic distance of x_j to the nearest point. Denote $B_j := B_\rho(x_j, r_j)$. Let ν_j stand for the harmonic measure on ∂B_j with respect to the point x_j (computed with respect to the ball B_j). We consider the random variables

$$Y_j = \int_{\partial B_j} X(z)\nu_j(dz)$$

(concerning the definition, an analogue of Remark 3.7 applies). By recalling (2.5), the covariance $C_X(z, w)$ is separately harmonic with respect to both of the variables. Since the balls B_j are disjoint, a standard limiting argument allows us to use the harmonicity of the Green's function to compute for $k \neq j$

$$\begin{aligned}\mathbb{E}\, Y_j Y_k &= \int_{\partial B_j}\Big(\int_{\partial B_k} C_X(z,w)\nu_k(dw)\Big)\nu_j(dz) = \int_{\partial B_j} C_X(z,x_k)\nu_j(dz)\\ &= C_X(x_j,x_k).\end{aligned} \tag{3.9}$$

We next observe that by the conformal invariance of the harmonic measure we have for any $h \in C(\partial B_j)$ that

$$\int_{\partial B_j} h(z)\nu_j(dz) = \fint_{\partial B_\rho(0,r_j)} h(\tau(w))|dw|,$$

where $\fint$ stands for the averaged integral and τ is a conformal self map of $\mathbb{D}$ that carries $B_\rho(0, r_j) \subset \mathbb{D}$ to B_j. By applying this formula and the conformal invariance of the GFF covariance we thus obtain

$$\begin{aligned}\mathbb{E}\, Y_j^2 &= \int_{\partial B_j}\Big(\int_{\partial B_j} C_X(z,w)\nu_j(dw)\Big)\nu_j(dz) \;=\; \fint_{\partial B_\rho(0,r_j)\times\partial B_\rho(0,r_j)} \log\Big|\frac{1-\overline{z}w}{z-w}\Big||dw||dz|\\ &= \fint_{\partial B_\rho(0,r_j)\times\partial B_\rho(0,r_j)} \log\Big|\frac{1}{z-w}\Big||dw||dz| \;=\; \log(1/r_j),\end{aligned} \tag{3.10}$$

where we noted the harmonicity of $\log|1 - \overline{z}w|$ and recalled the computation (3.6). We also used the fact that the standard radius of the pseudo hyperbolic ball centred at the origin is the same as the pseudo-hyperbolic one.

By performing our standard consideration of the expectation $\mathbb{E}\,\big|\sum_{k=1}^{N} q_j Y_j\big|^2$, in view of (3.9) we thus obtain the desired inequality with the right hand side

$$\frac{1}{2}\sum_{j=1}^{n} \log\left(\frac{1}{\frac{1}{2}\min_{k\neq j}\rho(x_j,x_k)}\right).$$

The conformal invariance of both the covariance and the pseudo hyperbolic metric ensures that the stated inequality with the above right hand side is actually true on any simply connected domain.

This yields the claim as we finally note that for any bounded domain there is a constant $a > 0$ so that $|z-w| \le a\rho(z,w)$. This last inequality is seen by noting that Koebe's estimate yields $|z-w| \le (2\mathrm{diam}(U))d_H(z,w) \approx \rho(z,w)$ for $\rho(z,w) \le 1/2$, and by boundedness of U this yields the claim. $\square$

Our goal in this section was to bound the moments of imaginary chaos. As noted already before, after Onsager's lemma the second ingredient we need for the upper bound is the following estimate. As the proof is a rather straightforward generalization of the 2-dimensional result in [45] it is given in the appendix.

Lemma 3.10. *Let $B(0,1)$ be the unit ball in $\mathbf{R}^d$. We have*

$$\int_{B(0,1)^N} \exp\Big(\frac{\beta^2}{2}\sum_{j=1}^N \log\frac{1}{\frac{1}{2}\min_{k\neq j}|x_j-x_k|}\Big)\,dx_1\dots dx_N \le c^N N^{N\frac{\beta^2}{2d}}$$

for some constant $c > 0$.

This lemma and Proposition 3.6 (ii) yield a bare uniform integrability statement which will be used to show that all the moments exist and that the formula (3.4) is indeed correct. This verifies the part of Theorem 1.3 which claims that $\mathbb{E}\,|\mu(f)|^k < \infty$ for all k.

Corollary 3.11. *Let K be a compact subset of U and assume that $x_1,\dots,x_N,y_1,\dots,y_N \in K$. Denote $z_1 = x_1,\ \dots,\ z_N = x_N,\ z_{N+1} = y_1,\ \dots,\ z_{2N} = y_N$. We have the uniform bound*

$$e^{-\beta^2\sum_{1\le j<k\le N}(C_{X_\varepsilon}(x_j,x_k)+C_{X_\varepsilon}(y_j,y_k))+\beta^2\sum_{1\le j,k\le N}C_{X_\varepsilon}(x_j,y_k)}$$
$$\le \exp\Big(\frac{\beta^2}{2}\sum_{j=1}^{2N}\log\frac{1}{\frac{1}{2}\min_{k\neq j}|z_j-z_k|}+cN^2\Big) =: \Xi_N(z_1,\dots,z_{2N})$$

for all $\varepsilon > 0$. Here the majorant Ξ_N depends on the subset K through the constant c, and is integrable over K^{2N}. A fortiori, the formula (3.4) *for the moments is valid for any $f \in C_c^\infty(U)$ under our standard assumptions* (2.2)*.*

Proof. We begin by writing out the moment $\mathbb{E}\,|\mu_\varepsilon(f)|^{2N}$ as a multiple integral

$$\begin{aligned}\mathbb{E}\,|\mu_\varepsilon(f)|^{2N} &= \int_{U^{2N}}\prod_{j=1}^N dx_j f(x_j)\prod_{j=1}^N dy_j\overline{f(y_j)}\,\mathbb{E}\,e^{i\beta\sum_{j=1}^N(X_\varepsilon(x_j)-X_\varepsilon(y_j))+\frac{\beta^2}{2}\sum_{j=1}^N(\mathbb{E}\,X_\varepsilon(x_j)^2+\mathbb{E}\,X_\varepsilon(y_j)^2)}\\ &= \int_{U^{2N}}\prod_{j=1}^N dx_j f(x_j)\prod_{j=1}^N dy_j\overline{f(y_j)}e^{-\beta^2\sum_{1\le j<k\le N}(C_{X_\varepsilon}(x_j,x_k)+C_{X_\varepsilon}(y_j,y_k))+\beta^2\sum_{1\le j,k\le N}C_{X_\varepsilon}(x_j,y_k)}\\ &\le \|f\|_\infty^{2N}\int_{(\operatorname{supp} f)^{2N}}\exp\Big(\frac{\beta^2}{2}\sum_{j=1}^{2N}\log\frac{1}{\frac{1}{2}\min_{k\neq j}|z_j-z_k|}+cN^2\Big)\,dz_1\dots dz_{2N}.\end{aligned}$$

Since the upper bound is independent of ε we may use the dominated convergence theorem to let $\varepsilon\to 0$ and deduce that the moments are finite and given by the right formula. $\square$

Lemma 3.10 combined with our versions of Onsager's inequality allows us to finally prove an upper bound for the moments of the purely imaginary chaos, verifying the moment bound portion of Theorem 1.3.

Theorem 3.12. *Assume that either $d = 2$ and $g \in C^2(U\times U)$, or d is arbitrary and $g \in H^{d+\varepsilon}_{loc}(U\times U)$ for some $\varepsilon > 0$. Then for every $N \ge 1$ and $f \in C_c^\infty(U)$ we have for μ from Theorem 1.1*

$$\mathbb{E}\,|\mu(f)|^{2N} \le \|f\|_\infty^{2N} C^N N^{\frac{\beta^2 N}{d}}$$

for some constant $C > 0$ (which may depend on the support of f).

Proof. To obtain the stated upper bounds, assume first that we are in the case $d = 2$ and $g \in C^2(U \times U)$. Then we may use Corollary 3.11 to infer

$$\mathbb{E}\,|\mu(f)|^{2N} = \int_{U^{2N}} \prod_{j=1}^{N} dx_j f(x_j) \prod_{j=1}^{N} dy_j \overline{f(y_j)} e^{-\beta^2 \sum_{1\le j<k\le N}(C_X(x_j,x_k)+C_X(y_j,y_k))+\beta^2\sum_{1\le j,k\le N} C_X(x_j,y_k)}$$

$$\le \|f\|_\infty^{2N} \int_{(\operatorname{supp} f)^{2N}} \exp\Big(\frac{\beta^2}{2}\sum_{j=1}^{2N} \log \frac{1}{\frac{1}{2}\min_{k\neq j}|z_j - z_k|} + cN\Big)\, dz_1 \dots dz_{2N},$$

where the last inequality is a consequence of part (1) of Proposition 3.6. The claim now follows from Lemma 3.10.

In the case where d is arbitrary and $g \in H^{d+\varepsilon}_{loc}(U \times U)$, we may by using compactness first cover $\operatorname{supp} f$ with a finite number of balls $B(a_1, \delta_1/2), \dots, B(a_m, \delta_m/2) \subset U$, where δ_ℓ are given by Theorem 3.8. Moreover, we can find a smooth partition of unity of non-negative functions $\eta_1, \dots, \eta_m$ such that $\operatorname{supp} \eta_\ell \subset B(a_\ell, \delta_\ell)$ and for any x in a small neighbourhood of $\operatorname{supp} f$ we have $\sum_{\ell=1}^m \eta_\ell(x) = 1$. Then

$$\mathbb{E}\,|\mu(f)|^{2N} = \mathbb{E}\,|\sum_{\ell=1}^{m} \mu(f\eta_\ell)|^{2N} \le m^{2N}\mathbb{E}\,\max_\ell(|\mu(f\eta_\ell)|^{2N}) \le m^{2N}\sum_{\ell=1}^{m}\mathbb{E}\,|\mu(f\eta_\ell)|^{2N},$$

and each summand may be approximated as in the previous case, replacing the use of Proposition 3.6 with Theorem 3.8. ☐

As the final component in the proof of Theorem 1.3, we record the following basic fact about the moments from Theorem 3.12 growing slowly enough for the moments to determine the law of μ.

Corollary 3.13. *Under the conditions of Theorem 3.12 all the exponential moments* $\mathbb{E}\, e^{\lambda|\mu(\varphi)|}$ *for* $\lambda \in \mathbf{R}$ *and* $\varphi \in C_c^\infty(U)$ *are finite and in particular the moments* $\mathbb{E}\,\mu(\varphi)^k\overline{\mu(\varphi)}^l$ *for* $k, l \ge 0$ *exist and they determine the distribution of* μ *as a random distribution in* $\mathcal{D}'(U)$.

Proof. As is standard, by linearity, the joint distribution of $(\mu(\phi_1), \dots \mu(\phi_m))$ for any number of test functions $\phi_j \in C_c^\infty(U)$ is determined as soon as the case of an arbitrary single test function, or $m = 1$ is known. This on the other hand, follows from Theorem 3.12, since the stated growth rate of the moments is well-known to be small enough to determine the distribution, see e.g. [33, Theorem 3.3.12]. Finally, the finiteness of exponential moments follows from expanding the exponential as a power series and using Theorem 3.12 coupled with a standard Jensen estimate. ☐

As mentioned, the proof of Theorem 1.3 now follows from combining Corollary 3.11, Theorem 3.12, and Corollary 3.13.

Asymptotics for moments in the case of the Gaussian Free Field (or more precisely for $g = 0$) have been proven in [45] by scaling and space partition arguments. Below we show how to slightly alter their method to deal with a general covariance $C_X(x, y)$ and obtain the following lower bounds for the moments. One should note that the main term in the estimate is the same as for the upper bound.

Proposition 3.14. *Let* $f \in C_c^\infty(U)$ *be non-negative and not identically zero. Then for* μ *from Theorem 1.1,*

$$\log \mathbb{E}\,|\mu(f)|^{2N} \ge \frac{\beta^2}{d} N \log N + \mathcal{O}(N).$$

Proof. By the assumption we may choose a cube $K \subset U$ so that $f \ge c_0 > 0$ on K. With a simple scaling and translation argument we may assume that $K = [0, 1]^d$ and $c_0 = 1$. Let us denote

$$Z_{\beta,2N}(\Omega) = \text{“}\mathbb{E}\,|\mu(\mathbf{1}_\Omega)|^{2N}\text{”} = \int_{\Omega^{2N}} \frac{\prod_{i,j=1}^N e^{\beta^2 C_X(x_i,y_j)}}{\prod_{1\le i<j\le N} e^{\beta^2 C_X(x_i,x_j)+\beta^2 C_X(y_i,y_j)}} \prod_{i=1}^N dx_i \prod_{j=1}^N dy_j,$$

for any measurable subset $\Omega \subset K$ and integer $N \ge 0$. Here we wrote “$\mathbb{E}\,|\mu(\mathbf{1}_\Omega)|^{2N}$” to indicate that we ignore the discussion about whether or not $\mathbf{1}_\Omega$ is a suitable test function, since it's only the integral we are interested in. Note that $\mathbb{E}\,|\mu(f)|^{2N} \ge Z_{\beta,2N}(K)$.

Assume that $0 \leq N_1 \leq N$ is an integer and write $N_2 = N - N_1$. Let also $\Omega_1, \Omega_2 \subset K$ be two measurable subsets (with positive $2N$-dimensional measure) satisfying $\Omega_1 \cap \Omega_2 = \emptyset$. Then the total integral defining $Z_{\beta,2N}(K)$, can be bounded from below by restricting to the subset of Ω^{2N} where precisely N_1 of both the x- and the y-variables are in Ω_1 and N_2 of them are in Ω_2. There are $\binom{N}{N_1}^2$ ways to choose the variables in this way and we find the following bound:

$$Z_{\beta,2N}(K) \geq \binom{N}{N_1}^2 Z_{\beta,2N_1}(\Omega_1) Z_{\beta,2N_2}(\Omega_2) \mathbb{E}_\nu e^{\beta^2 U} \geq \binom{N}{N_1}^2 Z_{\beta,2N_1}(\Omega_1) Z_{\beta,2N_2}(\Omega_2) e^{\beta^2 \mathbb{E}_\nu U},$$

where in the last step we used Jensen's inequality, and we have also introduced the following notation: ν is a probability measure on $\Omega_1^{2N_1} \times \Omega_2^{2N_2}$ of the form

$$\begin{aligned}\nu(dx^{(1)}, dy^{(1)}, dx^{(2)}, dy^{(2)}) = &\frac{1}{Z_{\beta,2N_1}(\Omega_1)} \frac{1}{Z_{\beta,2N_2}(\Omega_2)} \frac{\prod_{i,j=1}^{N_1} e^{\beta^2 C_X(x_i^{(1)}, y_j^{(1)})}}{\prod_{1\leq i<j\leq N_1} e^{\beta^2 C_X(x_i^{(1)}, x_j^{(1)}) + \beta^2 C_X(y_i^{(1)}, y_j^{(1)})}} \\ &\times \frac{\prod_{i,j=1}^{N_2} e^{\beta^2 C_X(x_i^{(2)}, y_j^{(2)})}}{\prod_{1\leq i<j\leq N_2} e^{\beta^2 C_X(x_i^{(2)}, x_j^{(2)}) + \beta^2 C_X(y_i^{(2)}, y_j^{(2)})}} dx^{(1)} dy^{(1)} dx^{(2)} dy^{(2)},\end{aligned}$$

where $dx^{(i)}$ and $dy^{(i)}$ denote the Lebesgue measure on $\Omega_i^{N_i}$, and we write

$$U = \log \frac{\prod_{i=1}^{N_1} \prod_{j=1}^{N_2} e^{C_X(x_i^{(1)}, y_j^{(2)}) + C_X(y_i^{(1)}, x_j^{(2)})}}{\prod_{i=1}^{N_1} \prod_{j=1}^{N_2} e^{C_X(x_i^{(1)}, x_j^{(2)}) + C_X(y_i^{(1)}, y_j^{(2)})}}.$$

We point out that the density of ν (as well as the domain of ν) is invariant under the transformation $x^{(2)} \leftrightarrow y^{(2)}$, but under this transformation U is mapped to $-U$, so we see that $\mathbb{E}_\nu U = 0$. We conclude that

$$Z_{\beta,2N}(K) \geq \binom{N}{N_1}^2 Z_{\beta,2N_1}(\Omega_1) Z_{\beta,2N_2}(\Omega_2),$$

or in other words

$$\frac{1}{[N!]^2} Z_{\beta,2N}(K) \geq \frac{1}{[N_1!]^2} Z_{\beta,2N_1}(\Omega_1) \frac{1}{[N_2!]^2} Z_{\beta,2N_2}(\Omega_2).$$

By induction, if $(\Omega_j)_{j=1}^k$ are non-empty disjoint positive measure subsets of K and $(N_j)_{j=1}^k$ are non-negative integers such that $N_1 + ... + N_k = N$, then

$$\frac{1}{[N!]^2} Z_{\beta,2N}(K) \geq \prod_{j=1}^{k} \frac{1}{[N_j!]^2} Z_{\beta,2N_j}(\Omega_j). \tag{3.11}$$

Let us now apply this inequality to the case where $k = \lceil N^{1/d} \rceil^d$, $N_j = 1$ for all $j = 1, \ldots, N$, $N_j = 0$ for $j = N+1, \ldots, k$, and Ω_j is a translate of $[0, \lceil N^{1/d} \rceil^{-1})^d$. This yields

$$\log Z_{\beta,2N}(K) \geq \log[N!]^2 + \sum_{i=1}^{N} \log Z_{\beta,2}(\Omega_i).$$

We now have for some vector $v_i \in [0,1)^d$

$$Z_{\beta,2}(\Omega_i) = \int_{[0,\lceil N^{1/d} \rceil^{-1})^{2d}} \frac{e^{\beta^2 g(v_i + x, v_i + y)}}{|x-y|^{\beta^2}} dx dy \ \geq \ e^{-\beta^2 \|g\|_{L^\infty(K)}} \lceil N^{1/d} \rceil^{\beta^2 - 2d} \int_{[0,1)^{2d}} \frac{1}{|x-y|^{\beta^2}} dx dy$$

so that

$$\log Z_{\beta,2N}(K) \geq 2N\log N - 2N + o(N) + \sum_{i=1}^{N}\Big[\Big(\frac{\beta^2}{d} - 2\Big)\log N + \mathcal{O}(1)\Big]$$
$$= \frac{\beta^2}{d}N\log N + \mathcal{O}(N). \qquad \square$$

As an application of the moment bounds we close this subsection by proving Theorem 1.4.

Proof of Theorem 1.4. Fix $\lambda > 1$. By Chebyshev's inequality and Theorem 3.12 we have for any $N \geq 1$ that

$$\log \mathbb{P}(|\mu(\varphi)| > \lambda) \leq \log \frac{\mathbb{E}\,|\mu(\varphi)|^{2N}}{\lambda^{2N}}$$
$$\leq \frac{\beta^2}{d}N\log(N) - 2N\log(\lambda) + cN$$

for some $c > 0$. Letting $N = \Big\lfloor \lambda^{\frac{2d}{\beta^2}} e^{-1-\frac{cd}{\beta^2}} \Big\rfloor$ and using the fact that the map $x \mapsto \frac{\beta^2}{d}x\log(x) - 2x\log(\lambda) + cx$ has Lipschitz constant of order 1 when $x \approx \lambda^{\frac{2d}{\beta^2}}$, we get

$$\log \mathbb{P}(|\mu(\varphi)| > \lambda) \leq \frac{\beta^2}{d}\lambda^{\frac{2d}{\beta^2}} e^{-1-\frac{cd}{\beta^2}} \frac{2d}{\beta^2}\log(\lambda) - \frac{\beta^2}{d}\lambda^{\frac{2d}{\beta^2}} e^{-1-\frac{cd}{\beta^2}}(1 + \frac{cd}{\beta^2})$$
$$- 2\lambda^{\frac{2d}{\beta^2}} e^{-1-\frac{cd}{\beta^2}}\log(\lambda) + c\lambda^{\frac{2d}{\beta^2}} e^{-1-\frac{cd}{\beta^2}} + O(1)$$
$$= -\frac{\beta^2}{d}\lambda^{\frac{2d}{\beta^2}} e^{-1-\frac{cd}{\beta^2}} + O(1).$$

To prove the lower bound, assume that there exist arbitrarily large numbers $\lambda > 0$ such that

$$\log \mathbb{P}(|\mu(\varphi)| > \lambda) \leq -\lambda^{\frac{2d}{\beta^2}+\varepsilon}$$

and fix some large enough $\lambda > 0$ (how large λ is needed will be implicitly determined during the proof). By assuming that λ is so large that $b := (\frac{1}{c})^{\frac{\beta^2}{2d}}\lambda^{1+\frac{\beta^2}{2d}\varepsilon} > \lambda$, we may compute for any $N \geq 1$ that

$$\mathbb{E}\,|\mu(\varphi)|^{2N} = 2N\Big(\int_0^\lambda + \int_\lambda^b + \int_b^\infty\Big)x^{2N-1}\mathbb{P}(|\mu(\varphi)| > x)\,dx$$
$$\leq 2Na\int_0^\lambda x^{2N-1}e^{-cx^{\frac{2d}{\beta^2}}}\,dx + 2Na\int_\lambda^b x^{2N-1}e^{-\lambda^{\frac{2d}{\beta^2}+\varepsilon}}\,dx + 2Na\int_b^\infty x^{2N-1}e^{-cx^{\frac{2d}{\beta^2}}}\,dx,$$

where we have used the bound $\mathbb{P}(|\mu(\varphi)| > x) \leq ae^{-cx^{2d/\beta^2}}$ (for some $c > 0$ and $a > 1$) coming from the first part of the proof, and applied the monotonicity of $\mathbb{P}(|\mu(\varphi)| > x)$ and the fact $a > 1$ when $x \in [\lambda, b]$. The length of the interval $[\lambda, b]$ is of the order $\lambda^{1+\frac{\beta^2}{2d}\varepsilon}$. By differentiation it is easy to check that the function $x \mapsto x^{2N-1}e^{-cx^{\frac{2d}{\beta^2}}}$ has a unique maximum at $x_0 = \big(\frac{\beta^2(2N-1)}{2dc}\big)^{\frac{\beta^2}{2d}}$. Fix some $\delta \in (0, \frac{\beta^2}{2d}\varepsilon)$. If we now choose $N \in [\frac{1}{2} + \frac{dc}{\beta^2}\lambda^{\frac{2d(1+\delta)}{\beta^2}}, 2(\frac{1}{2} + \frac{dc}{\beta^2}\lambda^{\frac{2d(1+\delta)}{\beta^2}})]$ to be an integer, (this is possible for large enough λ), then by this choice of N, the function $x \mapsto x^{2N-1}e^{-cx^{\frac{2d}{\beta^2}}}$ is increasing on the interval $[0, \lambda]$ (simply due to the fact that with this choice of N, we have $x_0 \geq \lambda^{1+\delta}$). The first integral is thus bounded by $2Na\lambda^{2N}e^{-c\lambda^{\frac{2d}{\beta^2}}}$. The second integral can be evaluated as $ae^{-\lambda^{\frac{2d}{\beta^2}+\varepsilon}}(b^{2N} - \lambda^{2N})$, and finally the third integral has the upper bound

$$2Na\int_b^\infty x^{2N-1}e^{-cx^{\frac{2d}{\beta^2}}}\,dx \leq 2Na\int_b^\infty \frac{b^{2N+1}e^{-cb^{\frac{2d}{\beta^2}}}}{x^2}\,dx \leq 2Nab^{2N+1}e^{-cb^{\frac{2d}{\beta^2}}},$$

where we have used the fact that $b > 1$ for large enough λ, and also that the unique maximum of $x \mapsto x^{2N+1}e^{-cx^{\frac{2d}{\beta^2}}}$, which is at the point $\big(\frac{\beta^2(2N+1)}{2dc}\big)^{\frac{\beta^2}{2d}}$, lies in $[\lambda, b]$ for large enough λ. Our choice of N shows that both $\lambda^{2d/\beta^2+\varepsilon}$ and b^{2d/β^2} grow quicker than $N^{1+\delta'}$ for some $\delta' > 0$, and hence the second and the third integrals converge to zero as $\lambda \to \infty$ as $\log b$ is of the order $\log N$. From the first integral we obtain that by increasing λ, we can find arbitrarily large integers $N = N(\lambda)$ for which

$$\mathbb{E}\,|\mu(\varphi)|^{2N} \lesssim e^{\frac{\beta^2 N}{d(1+\delta')}\log(N)}.$$

This contradicts the lower bound given by Proposition 3.14, and concludes the argument. □

3.3. **Regularity properties of imaginary chaos.** In this section we continue our study of analytic properties of imaginary chaos, namely we shall study to which classical function spaces imaginary chaos belongs – this corresponds to Theorem 1.2. We shall obtain essentially sharp results in Besov and Triebel–Lizorkin scales of function spaces, which include e.g. negative index Hölder spaces. As described in more detail in Section 2.2, this gives much more combined size and smoothness information on the chaos than obtained by just considering the Hilbert–Sobolev spaces $H^s(\mathbf{R}^d)$.

We start by proving that we are dealing with true generalised functions, instead of say honest functions or even complex measures. This is the first component of Theorem 1.2. Though this is an important fact, it seems not to have been proven in the literature before.

Theorem 3.15. *The imaginary chaos μ from Theorem 1.1 is almost surely not a complex measure.*

Proof. What the claim means is that the total variation of μ is almost surely infinite. To prove this, it is enough to find a sequence of smooth functions $(h_k)_{k\geq 1}$ on U such that almost surely $\sup_{k\geq 1}\|h_k\|_\infty \leq 1$ but $\sup_{k\geq 1}|\mu(h_k)| = \infty$. A suitable candidate turns out to be a subsequence of the random sequence

$$f_k(x) = e^{-i\beta X_{1/k}(x)}\psi(x),$$

where $X_{1/k}$ are standard mollifications of X, and the real-valued test function $\psi \in C_c^\infty(U)$ satisfies $\mathbf{1}_B \leq \psi(x) \leq \mathbf{1}_{2B}$, where $B = B(x_0, r_0)$ is a ball such that the double sized ball $2B := B(x_0, 2r_0)$ is compactly contained in U. The idea of the proof is to calculate $\mathbb{E}\,\mu(f_k)$ and $\mathbb{E}\,|\mu(f_k)|^2$ and argue by Paley–Zygmund that the total variation must be infinite with probability 1.

To simplify the notation, denote $g_\delta(x) = e^{-i\beta X_\delta(x)}\psi(x)$ so that $f_k(x) = g_{1/k}(x)$. Let us begin by computing $\mathbb{E}\,\mu(g_\delta)$. Using Proposition 3.1, we can pick a sequence $\varepsilon_n \searrow 0$ such that $\mu_{\varepsilon_n} \to \mu$ almost surely in say $H^{-d/2-1}(\mathbf{R}^d)$. Moreover, using the fact that $\mathbb{E}\,\|\mu_{\varepsilon_n}\|^2_{H^s(\mathbf{R}^d)}$ is bounded, which was part of the proof of Proposition 3.1, to justify a standard dominated convergence argument below, we see that

$$\begin{aligned}
\mathbb{E}\,\mu(g_\delta) &= \lim_{n\to\infty}\int_{2B}\mathbb{E}\,e^{i\beta X_{\varepsilon_n}(x)-i\beta X_\delta(x)}e^{\frac{\beta^2}{2}\mathbb{E}\,X_{\varepsilon_n}(x)^2}\psi(x)\,dx \\
&= \lim_{n\to\infty}\int_{2B}e^{-\frac{\beta^2}{2}\mathbb{E}\,X_\delta(x)^2+\beta^2\mathbb{E}\,X_{\varepsilon_n}(x)X_\delta(x)}\psi(x)\,dx \\
&= \int_{2B}e^{-\frac{\beta^2}{2}\mathbb{E}\,X_\delta(x)^2+\beta^2\mathbb{E}\,X(x)X_\delta(x)}\psi(x)\,dx =: A_\delta,
\end{aligned}$$

where $\mathbb{E}\,X(x)X_\delta(x) = \lim_{\varepsilon\to 0}\mathbb{E}\,X_\varepsilon(x)X_\delta(x)$. Note that by Lemma 2.8 we have $A_\delta \gtrsim \delta^{-\frac{\beta^2}{2}}$.

To compute $\mathbb{E}\,|\mu(g_\delta)|^2$, we argue in a similar way, but now L^2-boundedness is not sufficient for us to conclude. The remedy comes from Proposition 3.6(ii) which can be used to check that say $\sup_{n\geq 1}\mathbb{E}\,\|\mathbf{1}_{2B}\mu_{\varepsilon_n}\|^6_{H^{-d/2-1}(\mathbf{R}^d)} < \infty$. In turn, by the smoothness of the covariance C_{X_δ} one easily verifies that $\mathbb{E}\,\|g_\delta\|^6_{H^{d/2+1}(\mathbf{R}^d)} < \infty$ for each fixed $\delta > 0$. Thus one finds that we can interchange the order of the limit and integration and we now obtain

$$
\begin{aligned}
\mathbb{E}\,|\mu(g_\delta)|^2 &= \lim_{n\to\infty} \int_{2B}\int_{2B} \mathbb{E}\, e^{i\beta X_{\varepsilon_n}(x) - i\beta X_{\varepsilon_n}(y) - i\beta X_\delta(x) + i\beta X_\delta(y)} \cdot \\
&\qquad\qquad e^{\frac{\beta^2}{2}\mathbb{E}\, X_{\varepsilon_n}(x)^2 + \frac{\beta^2}{2}\mathbb{E}\, X_{\varepsilon_n}(y)^2} \psi(x)\psi(y)\, dx\, dy \\
&= \lim_{n\to\infty} \int_{2B}\int_{2B} e^{-\frac{\beta^2}{2}\mathbb{E}\, X_\delta(x)^2 - \frac{\beta^2}{2}\mathbb{E}\, X_\delta(y)^2 + \beta^2 \mathbb{E}\, X_{\varepsilon_n}(x)X_\delta(x) + \beta^2\mathbb{E}\, X_{\varepsilon_n}(y)X_\delta(y)} \psi(x)\psi(y) \cdot \\
&\qquad\qquad e^{\beta^2\mathbb{E}\, X_\delta(x)X_\delta(y) + \beta^2\mathbb{E}\, X_{\varepsilon_n}(x)X_{\varepsilon_n}(y) - \beta^2\mathbb{E}\, X_{\varepsilon_n}(x)X_\delta(y) - \beta^2\mathbb{E}\, X_\delta(x)X_{\varepsilon_n}(y)}\, dx\, dy \\
&= \int_{2B}\int_{2B} e^{-\frac{\beta^2}{2}\mathbb{E}\, X_\delta(x)^2 - \frac{\beta^2}{2}\mathbb{E}\, X_\delta(y)^2 + \beta^2\mathbb{E}\, X_\delta(x)X(x) + \beta^2\mathbb{E}\, X_\delta(y)X(y)} \psi(x)\psi(y) \cdot \\
&\qquad\qquad e^{\beta^2\mathbb{E}\, X_\delta(x)X_\delta(y) + \beta^2\mathbb{E}\, X(x)X(y) - \beta^2\mathbb{E}\, X_\delta(x)X(y) - \beta^2\mathbb{E}\, X_\delta(y)X(x)}\, dx\, dy \\
&=: B_\delta.
\end{aligned}
$$

Our aim is to show that $\lim_{\delta\to 0} \frac{A_\delta^2}{B_\delta} = 1$. Let

$$
a_\delta(x) := \delta^{\beta^2/2} e^{-\frac{\beta^2}{2}\mathbb{E}\, X_\delta(x)^2 + \beta^2\mathbb{E}\, X_\delta(x)X(x)} \psi(x)
$$

and

$$
b_\delta(x,y) := e^{\beta^2\mathbb{E}\, X_\delta(x)X_\delta(y) + \beta^2\mathbb{E}\, X(x)X(y) - \beta^2\mathbb{E}\, X_\delta(x)X(y) - \beta^2\mathbb{E}\, X_\delta(y)X(x)}.
$$

Then we have

$$
\frac{B_\delta}{A_\delta^2} = \frac{\int_{2B}\int_{2B} a_\delta(x)a_\delta(y)b_\delta(x,y)\, dx\, dy}{\Big(\int_{2B} a_\delta(x)\, dx\Big)^2} = 1 + \frac{\int_{2B}\int_{2B} a_\delta(x)a_\delta(y)(b_\delta(x,y)-1)\, dx\, dy}{\Big(\int_{2B} a_\delta(x)\, dx\Big)^2}. \tag{3.12}
$$

By Lemma 2.8 we know that $a_\delta(x)$ is bounded both from above and away from 0, uniformly in δ and x. Moreover, $b_\delta(x,y)$ has an integrable majorant of the form $C|x-y|^{-\beta^2}$ for some $C > 0$, and it converges to 1 pointwise. Thus by the dominated convergence theorem the right hand side of (3.12) tends to 1 as $\delta \to 0$, as desired.

Since $\mathbb{E}\,|\mu(f_k)| \ge \mathbb{E}\,\mu(f_k)$, the Paley–Zygmund inequality shows that we have

$$
\mathbb{P}(|\mu(f_k)| > \theta\mathbb{E}\,|\mu(f_k)|) \ge (1-\theta)^2 \frac{(\mathbb{E}\,\mu(f_k))^2}{\mathbb{E}\,|\mu(f_k)|^2}
$$

for any $\theta \in (0,1)$. Choosing $\theta = (\mathbb{E}\,|\mu(f_k)|)^{-\varepsilon}$ for some $\varepsilon > 0$ we thus see that

$$
\mathbb{P}(|\mu(f_k)| > (\mathbb{E}\,|\mu(f_k)|)^{1-\varepsilon}) \ge (1 - (\mathbb{E}\,|\mu(f_k)|)^{-\varepsilon})^2 \frac{A_{1/k}^2}{B_{1/k}} \to 1
$$

as $k \to \infty$. As we noted above that $A_\delta \gtrsim k^{(1-\varepsilon)\frac{\beta^2}{2}}$, this implies that

$$
\mathbb{P}\big(|\mu(f_k)| \ge Ck^{(1-\varepsilon)\frac{\beta^2}{2}} \text{ for infinitely many } k\big) = 1 \tag{3.13}
$$

for some constant $C > 0$. This provides us with the desired subsequence h_k and proves the claim. We note that for our purposes here one could have chosen for instance $\varepsilon = 1/2$, but we stated (3.13) for later use in the proof of Theorem 3.16 below. □

The following general result can be used to show that the imaginary chaos belongs to $C^s_{loc}(U)$ or $H^s_{loc}(U)$[17] for indices $s < -\beta^2/2$, and this range is essentially optimal. Moreover, the optimality is not due to some special boundary effects since it is shown using localisations that lie compactly inside the domain U. This is the second part of Theorem 1.2.

Theorem 3.16. *Assume that $\beta \in (0, \sqrt{d})$ and fix $1 \le p, q \le \infty$. Moreover, let X be a log-correlated field satisfying our basic assumptions* (2.1) *and* (2.2). *Let μ be the imaginary chaos given by Theorem 1.1. Then the following are true.*

[17] The definition of localised functions spaces with subscript *loc* was given in Subsection 2.2. We also recall that for general $s \in \mathbf{R}$, the interpretation of C^s is $B^s_{\infty,\infty}$ – see again Section 2.2.

(i) *We have almost surely* $\mu \in B^s_{p,q,loc}(U)$ *when* $s < -\frac{\beta^2}{2}$, *and* $\mu \notin B^s_{p,q,loc}(U)$ *for* $s > -\frac{\beta^2}{2}$.
(ii) *Assume moreover that* $g \in L^\infty(U \times U)$ *or that* X *is the* 2d *GFF with zero boundary conditions. Then almost surely* $\mu \in B^s_{p,q}(\mathbf{R}^d)$ *when* $s < -\frac{\beta^2}{2}$.
(iii) *Analogous statements hold for the Triebel spaces in the case* $p, q \in [1, \infty)$.

Proof. (i). Fix $\psi \in C_c^\infty(U)$, and denote the support of ψ by K so that K is a compact subset of U. In view of the inclusions (2.17),(2.18) and the embedding (2.19), in order to prove the claim it is enough to establish that for any $s < -\beta^2/2$ and for arbitrary large positive integers n it holds that $\psi\mu \in B^s_{2n,2n}(\mathbf{R}^d)$ almost surely.

We fix a large n and compute a suitable moment of the Besov-norm as follows

$$\mathbb{E}\, \|\psi\mu\|^{2n}_{B^s_{2n,2n}} = \mathbb{E} \sum_{j=0}^\infty 2^{2nsj} \int_{\mathbf{R}^d} |((\psi\mu) * \phi_j)(x)|^{2n}\, dx,$$

where the ϕ_j:s are as in the discussion leading to (2.16). By Proposition 3.6(ii), and using the fact that the integrand is invariant under permutations of the whole set of variables $x_1, \dots, x_n, y_1, \dots, y_n$, we see that it is enough to check that

$$\sum_{j=0}^\infty 2^{2nsj} \int_{\mathbf{R}^d} \int_{K^{2n}} \frac{|\phi_j(x-x_1)\dots\phi_j(x-x_n)\phi_j(x-y_1)\dots\phi_j(x-y_n)|}{|x_1-y_1|^{\beta^2}\dots|x_n-y_n|^{\beta^2}}\, dx_1 \dots dx_n\, dy_1 \dots dy_n\, dx$$

is finite. As for $j \geq 1$, the functions ϕ_j are built from ϕ_1, we consider separately $j = 0$ and $j \geq 1$. The summand for $j = 0$ is clearly finite (by compact support and the fact that $\beta^2 < d$). For $j \geq 1$, pick a ball B centered at the origin such that $K \subset B$. For the rest of the sum the change of variables $x_k \mapsto 2^{-j}x_k$, $y_k \mapsto 2^{-j}y_k$ and $x \mapsto 2^{-j}x$ yields the upper bound

$$\sum_{j=1}^\infty 2^{2nsj+nj\beta^2-jd} \int_{\mathbf{R}^d} \left(\int_{(2^jB)\times(2^jB)} \frac{|\phi_1(x-x_1)\phi_1(x-y_1)|}{|x_1-y_1|^{\beta^2}}\, dx_1\, dy_1 \right)^n dx,$$

where we have used the fact that $\phi_j(x) = 2^{dj}\phi_1(2^jx)$. Comparing with our statement, we see that it is enough to check that

$$\int_{\mathbf{R}^d} 2^{-jd} \left(\int_{(2^jB)\times(2^jB)} \frac{|\phi_1(x-x_1)\phi_1(x-y_1)|}{|x_1-y_1|^{\beta^2}}\, dx_1\, dy_1 \right)^n dx$$

is uniformly bounded in j. Notice that for $x \in 2^{j+1}B$ we have

$$\begin{aligned} &\int_{(2^jB)\times(2^jB)} \frac{|\phi_1(x-x_1)\phi_1(x-y_1)|}{|x_1-y_1|^{\beta^2}}\, dx_1\, dy_1 \\ &\lesssim \int_{\mathbf{R}^d} \frac{1}{(1+|x-x_1|^{2d})(1+|x-y_1|^{2d})|x_1-y_1|^{\beta^2}}\, dx_1\, dy_1 \leq c', \end{aligned}$$

as the integral is constant in x. Moreover, for $x \notin 2^{j+1}B$ we have

$$\begin{aligned} &\int_{x\notin 2^{j+1}B} 2^{-jd} \left(\int_{(2^jB)\times(2^jB)} \frac{|\phi_1(x-x_1)\phi_1(x-y_1)|}{|x_1-y_1|^{\beta^2}} \right)^n dx \\ &\leq \int_{x\notin 2^{j+1}B} 2^{-jd} \left(\int_{(2^jB)\times(2^jB)} \frac{1}{(1+|x-x_1|^{2d})(1+|x-y_1|^{2d})|x_1-y_1|^{\beta^2}}\, dx_1\, dy_1 \right)^n dx \\ &\leq \int_{x\notin 2B} 2^{(2jd-j\beta^2)n} \left(\int_{B\times B} \frac{1}{(1+2^{2dj}|x-x_1|^{2d})(1+2^{2dj}|x-y_1|^{2d})|x_1-y_1|^{\beta^2}}\, dx_1\, dy_1 \right)^n dx, \\ &\lesssim \int_{x\notin 2B} \frac{2^{n(-2jd-j\beta^2)}}{|x|^{4dn}} \left(\int_{B\times B} \frac{1}{|x_1-y_1|^{\beta^2}}\, dx_1\, dy_1 \right)^n dx \end{aligned}$$

which goes to 0 as $j \to \infty$. This concludes the proof of $\psi\mu \in B^s_{2n,2n}(\mathbf{R}^d)$ almost surely, and thus by our discussion at the beginning of the proof, this implies that for $s < -\frac{\beta^2}{2}$, $\mu \in B^s_{p,q,loc}(U)$ almost surely.

We then turn to the converse direction. In this case one deduces from (2.17),(2.18) and (2.19) that it is enough to verify for any fixed $s < \frac{\beta^2}{2}$ that almost surely $\psi\mu \notin B^{-s}_{1,1}$. From (3.13) we know that if we let $f_k(x) = \psi(x)e^{-i\beta X_{1/k}(x)}$ with ψ as in the proof of Theorem 3.15, then for any $\delta > 0$ there exists a deterministic constant C and a stochastic sequence $n_k \to \infty$ such that $|\mu(f_{n_k})| \geq C n_k^{\frac{\beta^2}{2}-\delta}$ with probability one. By the duality of $B^{-s}_{1,1}$ and $B^s_{\infty,\infty}$ we thus have

$$\|\mu\|_{B^{-s}_{1,1}} \geq \frac{C n_k^{\frac{\beta^2}{2}-\delta}}{\|f_{n_k}\|_{B^s_{\infty,\infty}}},$$

and hence it is enough to show that for all fixed $\delta > 0$ the inequality $\|f_n\|_{B^s_{\infty,\infty}} \leq n^{s+2\delta}$ holds almost surely for large enough n. We will prove this bound first in the case when $s < 1$. The norm $\|f_k\|_{B^s_{\infty,\infty}}$ is equivalent to the Hölder norm of f_k, and since $t \mapsto e^{-i\beta t}$ is Lipschitz, it is enough to consider the C^s-norm of $X_{1/k}$. In order to bound this, we note first that for a fixed $\delta \in (0, (1-s)/2)$

$$\|X_{1/n}\|_{C^s} \sim c\|I^\delta X_{1/n}\|_{C^{s+\delta}} \sim \|(I^\delta X)_{1/n}\|_{C^{s+\delta}},$$

where I^δ is the standard lift operator (2.20) (see the definition in Subsection 2.2) $I^\delta \colon C^s \to C^{s+\delta}$, and $c > 0$ is a constant. By Lemma 2.5 we have $I^\delta X \in C^{\delta/2}$ almost surely, and thus by Fernique's theorem

$$\mathbb{E} \exp\left(a\|I^\delta X\|^2_{C^{\delta/2}}\right) < \infty$$

for some $a > 0$. Moreover, we may compute directly from the definition of a convolution that

$$\begin{aligned} |(I^\delta X)_{1/n}(x) - (I^\delta X)_{1/n}(y)| &\leq \|I^\delta X\|_\infty \int |\eta_{1/n}(x-u) - \eta_{1/n}(y-u)|\, du \\ &\leq b\|I^\delta X\|_\infty \min(1, n|x-y|) \end{aligned} \tag{3.14}$$

for some constant $b > 0$. Thus

$$\begin{aligned} \|(I^\delta X)_{1/n}\|_{C^{s+\delta}} &\leq b \sup_{|x-y|\leq 1} |x-y|^{-s-\delta} \min(1, n|x-y|)\|I^\delta X\|_\infty + \|(I^\delta X)_{1/n}\|_\infty \\ &\lesssim (n^{s+\delta} + 1)\|I^\delta X\|_{C^{\delta/2}}. \end{aligned}$$

By the Fernique bound we have

$$\mathbb{P}(\|(I^\delta X)_{1/n}\|_{C^{s+\delta}} > n^{s+2\delta}) \leq \mathbb{P}((n^{s+\delta}+1)\|I^\delta X\|_{C^{\delta/2}} \geq n^{s+2\delta}) \leq e^{-b' n^\delta}$$

for some constant $b' > 0$. Finally, by Borel–Cantelli $\|f_n\|_{C^s} \leq n^{s+2\delta}$ for all large enough $n \geq n(\omega)$. This is precisely what we set out to prove, so we are done in the $s < 1$ case.

In the case of $s \geq 1$, we may actually choose $s > 1$ and we need to get an estimate for the Hölder norm of the derivatives of $X_{1/n}$. This is obtained by applying estimates like (3.14) by replacing the test function η by its derivatives. We leave the details for the reader.

(ii) The proof is identical to that in case (i) as one invokes Proposition 3.9.

(iii) The claims for the Triebel–Lizorkin spaces follow easily from those for the Besov spaces by employing the embeddings (2.17). ☐

Combining Theorem 3.15 and Theorem 3.16 yields Theorem 1.2, so this concludes our study of regularity properties of imaginary chaos and we turn to what we refer to as universality properties.

3.4. **Universality properties.** The goal of this section is to study the following question: For which periodic functions H can we make sense of $H(X)$ (through a suitable regularization and renormalization procedure) when X is a log-correlated field? To give an intuitive answer to this question, let us assume that H is a $2\pi/\beta$-periodic[18] function and let us expand $H(X_n)$ as a Fourier series $H(X_n(x)) = \sum_{k\in\mathbf{Z}} H_k e^{ik\beta X_n(x)}$. Now if $H_0 \neq 0$, we would expect from Proposition 3.1 that $H(X_n(x)) \to H_0$ as $n \to \infty$. If on the other hand $H_0 = 0$, and β is small enough, then one would expect that multiplying by $e^{\frac{\beta^2}{2}\mathbb{E}[X_n(x)^2]}$ and letting $n \to \infty$ would pick out the $k = \pm 1$-terms and yield $H_1 e^{i\beta X(x)} + H_{-1}e^{-i\beta X(x)}$. If $H_{\pm 1} = 0$ and β is small enough, one would expect convergence to $H_2 e^{2i\beta X(x)} + H_{-2}e^{-2i\beta X(x)}$ and so on. To make this argument rigorous, one needs to control the contribution of the higher Fourier modes. For simplicity we shall assume from now on that H is real, even, $H_0 = 0$, and $H_1 \neq 0$, though these assumptions can be relaxed, see Remark 3.19 below.

Before proceeding, let us address a technicality that might concern a careful reader. If H is not very regular, say just measurable instead of continuous, one might worry whether or not $\int H(X_n(x))\varphi(x)dx$ is a well defined random variable. That is, if $\widetilde{H} = H$ Lebesgue almost everywhere, do we have $\int H(X_n(x))\varphi(x)dx = \int \widetilde{H}(X_n(x))\varphi(x)dx$ almost surely? To see that this is the case, note that if X_n is a centered Gaussian field with continuous realisations on the bounded domain $U \subset \mathbf{R}^d$, pointwise non-degenerate (i.e. $\mathbb{E}\, X_n(x)^2 > 0$ for each $x \in U$), and $H : \mathbf{R} \to \mathbf{R}$ is a locally bounded function, then for any bounded compactly supported measurable function φ, the evaluation

$$Y := \int_U H(X_n(x))\varphi(x)dx$$

is well-defined as a random variable. Indeed, we may choose Borel measurable representatives for the functions H and φ, and it follows that $(x,\omega) \mapsto H(X_n(x,\omega))\varphi(x)$ is jointly measurable. Moreover, given another Borel measurable representative $\widetilde{H}$, one has a.s. $H(X_n(x,\omega)) = \widetilde{H}(X_n(x,\omega))$ for almost every $x \in U$ by Fubini's theorem and the fact that Gaussians have continuous density on $\mathbf{R}^d$. Hence moving to $\widetilde{H}$ does not change the value of Y, and we do not need to assume much regularity from H to pose a meaningful question.

In what follows we assume again that (X_n) are standard convolution approximations of our log-correlated field X on the domain $U \subset \mathbf{R}^d$. To be more precise, we write $X_n := X * \eta_{c_n}$ for some sequence $c_n \to 0$ as in Lemma 2.8, and we recall that the covariance $C_n(x,y) := C_{X_n}(x,y)$ satisfies for any compact subset $K \subset U$, that there exists a $M = M(K)$ such that

$$\Big|C_n(x,y) - \log\big(\frac{1}{\max(c_n, |x-y|)}\big)\Big| \le M \quad \text{for all} \quad x,y \in K, \tag{3.15}$$

as $n \to \infty$.

The following lemma is instrumental in controlling the contribution of higher order Fourier modes. We are able to obtain a result for a slightly larger class of functions H when specializing to two dimensions and assuming some further regularity from g, and for this reason, we also prove a slightly stronger version of our control of higher Fourier modes in the case of $d = 2$.

Lemma 3.17. *Let X be a log-correlated field satisfying assumptions* (2.1) *and* (2.2) *and let (X_n) be a convolution approximation of it as described above. Assume that $\beta \in (0, \sqrt{d})$ and $\varphi \in L^\infty(U)$ has compact support. Denote for $k \in \mathbf{Z}$*

$$Y_k := \int_U \varphi(x) e^{\frac{1}{2}\beta^2 C_n(x,x)} e^{ik\beta X_n(x)}dx.$$

(i) *For all integers k with $|k| \ge 2$ it holds that*

$$\mathbb{E}\,|Y_k|^2 \lesssim c_n^\alpha \|\varphi\|_{L^\infty}^2, \tag{3.16}$$

where c_n is as in (3.15)*, $\alpha = \min(3\beta^2, d - \beta^2)$, and in the special case $\beta = \frac{1}{2}\sqrt{d}$ the factor on the right hand side must be replaced by $c_n^{3\beta^2}\log\frac{1}{c_n}$. The bounds are uniform in k.*

[18]This is simply a notationally convenient way to write the arbitrary period of the function as it will work well with the notation we have used previously.

(ii) *Assume that $d = 2$, $g \in C^2(U\times U)$ and assume that the bump function η used to define the convolution approximations X_n is additionally non-negative, radially decreasing and symmetric, so that $\eta(0) > 0$. Moreover, assume that the term g in the covariance satisfies $g(x,x) = g(y,y)$ for all $x, y \in U$. Then for all integers ℓ, k with $|\ell|, |k| \geq 4/\beta$ and for n large enough it holds that*

$$|\mathbb{E}\, Y_k\overline{Y_\ell}| \lesssim \frac{(e^{2M}c_n)^{-\beta^2+2+\frac{\beta^2}{8}(\ell-k)^2}}{(|\ell|\vee|k|)^2}\|\varphi\|_{L^\infty}^2. \tag{3.17}$$

Proof. (i) We may assume that $\|\varphi\|_{L^\infty} = 1$ and denote $K := \mathrm{supp}(\varphi) \subset U$, so that K is compact. A direct computation yields the upper bound

$$\mathbb{E}\,|Y_n|^2 \leq I_n := \|\varphi\|_{L^\infty}^2 \int_{K\times K} \exp\Big(\beta^2k^2C_n(x,y) - \frac{1}{2}\beta^2(k^2-1)(C_n(x,x)+C_n(y,y)\Big)dxdy.$$

In the range $k^2\beta^2 < d$ the term $\exp(\beta^2k^2C_n(x,y))$ is uniformly integrable in n, As $|C_n(x,x) - \log(1/c_n)| \lesssim 1$ for all x we infer that $I_n \lesssim c_n^{(k^2-1)\beta^2}$. In the case $k^2\beta^2 = d$ we obtain a similar bound where one just adds the extra factor

$$\int_{U\times U} e^{dC_n(x,y)}dxdy \sim \int_{|x-y|\leq c_n} c_n^{-d} + \int_{|x-y|\geq c_n} |x-y|^{-d} \sim \log(1/c_n).$$

In the generic situation $k^2\beta^2 > d$ we observe first that due to the covariance inequality

$$C_n(x,y) \leq \frac{1}{2}(C_n(x,x)+C_n(y,y)), \tag{3.18}$$

the integrand in I_n is upper bounded by $\exp(\beta^2C_n(x,y))$. We use this estimate in the part of the product domain where $|x-y| \leq e^{2M}c_n$ and note that for the remaining values $|x-y| > e^{2M}c_n$, where M is from (3.15), we have

$$\begin{aligned} &\beta^2k^2C_n(x,y) - \frac{1}{2}\beta^2(k^2-1)(C_n(x,x)+C_n(y,y)) \\ \leq\ &\beta^2k^2(\log(|x-y|^{-1}) + M - \beta^2(k^2-1)(\log(1/c_n) - M) \\ \leq\ &\beta^2k^2\log\big(\big|(x-y)e^{-2M}\big|^{-1}\big) - \beta^2(k^2-1)\log(1/c_n). \end{aligned}$$

Thus

$$\begin{aligned} I_n &\lesssim \int_{|x-y|\leq e^{2M}c_n} |x-y|^{-\beta^2}dxdy + c_n^{\beta^2(-1+k^2)}\int_{|x-y|>e^{2M}c_n} |(x-y)e^{-2M}|^{-k^2\beta^2}dxdy \\ &\lesssim c_n^{-\beta^2+d}, \end{aligned}$$

where in the latter integral one performs a change of variables $(x,y) = (e^{2M}x', e^{2M}y')$. The claim follows by combining our estimates for different values of k.

(ii) We use the same notation as in the proof of part (i). Consider first the case where ℓ and k have the same sign, so that we may assume $k, \ell > 0$. We claim first that given any constant $A > 0$, for points $x, y \in K$ it holds with a constant $\delta = \delta(K, A, g) > 0$ and large enough $n \geq n_0(K, A, g)$ that

$$C_n(x,y) \leq C_n(x,x) - \delta\big(|x-y|/c_n\big)^2 \quad \text{if} \quad |x-y| \leq Ac_n. \tag{3.19}$$

This auxiliary result will be used later on in the proof. In order to verify (3.19), we fix $y_0 \in K$ and note that

$$\begin{aligned} C_n(x,y_0) &= \big(\widetilde{\eta}_{c_n} * \log(|\cdot|^{-1})\big)(x-y_0) + \big((\eta_{c_n}\otimes\eta_{c_n}) * g\big)(x,y_0) \\ &=: V_n(x) + W_n(x) \end{aligned}$$

where $\widetilde{\eta} := \eta * \eta$. Since $C_n(x,x)$ is independent of x, it follows from the covariance inequality (3.18) that $V_n(x) + W_n(x)$ has a maximum at $x = y_0$ and we have $\nabla(V_n + W_n)(y_0) = 0$. By symmetry considerations $\nabla V_n(y_0) = 0$, whence also $\nabla W_n(y_0) = 0$. Since D^2W_n is bounded in any compact subdomain of U, uniformly in n, we may easily infer the uniform bound

$$W_n(x) - W_n(y_0) \leq C|x-y_0|^2,$$

valid uniformly for $(x,y_0)\in K\times K$, and $C=C(K)$. On the other hand, the function $V_0:=\big(\widetilde\eta_1 * \log(|\cdot|^{-1})\big)(x-y_0)$, defined for all $x\in\mathbf{R}^2$, obtains its unique maximum at the point y_0 by (an integral version of) the Hardy–Littlewood rearrangement inequality, and as the logarithm yields the fundamental solution of the Laplacian in the plane, we have $\Delta V_0(y_0)=-2\pi\widetilde\eta_1(0)<0$. As V_0 is radial with respect to y_0 it follows easily that for any given $A\geq 1$ there is $\delta=\delta(A)>0$ such that $V_0(x)-V_0(y_0)\leq -2\delta|x-y_0|^2$ for $|x-y_0|\leq A$. Then the scaling properties of the logarithm yield that

$$V_n(x)-V_n(y_0)\leq -2\delta(|x-y_0|/c_n)^2 \quad\text{for}\quad |x-y_0|\leq Ac_n.$$

By combining this with our previous estimate for $W_n(x)-W_n(y_0)$ the inequality (3.19) follows for large enough n.

We now move to actually estimating $\mathbb{E}\,|Y_k\overline{Y_\ell}|$. We recall that $K\subset U$ is the topological support of φ and compute

$$\begin{aligned}
\big|\mathbb{E}\,Y_k\overline{Y_\ell}\big| &= \left|\int_{K\times K}\varphi(x)\overline{\varphi(y)}\exp\left[\ell k\beta^2 C_n(x,y)-\frac{\beta^2}{2}\big((\ell^2-1)C_n(x,x)+(k^2-1)C_n(y,y)\big)\right]dxdy\right|\\
&\lesssim c_n^{-\beta^2}\int_{K\times K}\exp\left[\ell k\beta^2 C_n(x,y)-\frac{\beta^2}{2}\big(\ell^2C_n(x,x)+k^2C_n(y,y)\big)\right]dxdy\\
&= c_n^{-\beta^2}\int_{\{|x-y|\leq e^{2M}c_n\}\cap K\times K} e^{\ell k\beta^2 C_n(x,y)-\frac{\beta^2}{2}\big(\ell^2C_n(x,x)+k^2C_n(y,y)\big)}dxdy\\
&\quad + c_n^{-\beta^2}\int_{\{|x-y|> e^{2M}c_n\}\cap K\times K} e^{\ell k\beta^2 C_n(x,y)-\frac{\beta^2}{2}\big(\ell^2C_n(x,x)+k^2C_n(y,y)\big)}dxdy\\
&=: I_n^1+I_n^2.
\end{aligned}$$

In the set $\{|x-y|>e^{2M}c_n\}\cap K\times K$ we may estimate

$$\begin{aligned}
&\ell k\beta^2 C_n(x,y)-\frac{\beta^2}{2}\big(\ell^2C_n(x,x)+k^2C_n(y,y)\big)\\
\leq\ & \ell k\beta^2(\log(|x-y|^{-1})+M)-\frac{\beta^2}{2}(\ell^2+k^2)(\log(1/c_n)-M)\\
\leq\ & \ell k\beta^2\big(\log\big(|e^{2M}|x-y|^{-1}|\big)-M\big)-\frac{\beta^2}{2}(\ell^2+k^2)(\log(1/c_n)-M)\\
\leq\ & \ell k\beta^2\log\big(|e^{2M}|x-y|^{-1}|\big)-\frac{\beta^2}{2}(\ell^2+k^2)\log(1/c_n)+\frac{M\beta^2}{2}(\ell-k)^2.
\end{aligned}$$

We denote $M':=e^M$ and perform the change of variables $u=(x-y)/(M')^2$, $v=(x+y)/(M')^2$. After integration first with respect to the variable v it follows that

$$\begin{aligned}
I_n^2\ &\lesssim\ c_n^{-\beta^2+\frac{\beta^2}{2}(\ell^2+k^2)}M'^{\beta^2(\ell-k)^2/2}\int_{|u|\geq c_n}|u|^{-k\ell\beta^2}du\\
&\lesssim\ \frac{(c_nM')^{-\beta^2+2+\frac{\beta^2}{2}(\ell-k)^2}}{k\ell\beta^2-2}\lesssim\frac{(c_nM')^{-\beta^2+2+\frac{\beta^2}{2}(\ell-k)^2}}{\ell k}\lesssim\frac{(c_nM')^{-\beta^2+2+\frac{\beta^2}{4}(\ell-k)^2}}{(\ell\vee k)^2},
\end{aligned}$$

where in the second last inequality we used the fact that $\frac14 k\ell\beta^2\geq 2$, which follows from our assumption that $|k|,|\ell|\geq 4/\beta$. In turn, the last inequality follows by noting that we may assume $\ell>k$, and by considering separately the cases $\ell\geq 2k$ and $2k>\ell$. Naturally, we need to assume that n is large enough so that, say, $c_n^{-1}>2M'$.

Next, for I_n^1 we have $|x-y|\leq e^{2M}c_n$. Using (3.19) yields that

$$\begin{aligned}
&\ell k\beta^2 C_n(x,y)-\frac{\beta^2}{2}\big(\ell^2C_n(x,x)+k^2C_n(y,y)\big)\\
\leq\ & \ell k\beta^2(C_n(x,x)-\delta(|x-y|/c_n)^2)-\frac{\beta^2}{2}\big(\ell^2+k^2\big)C_n(x,x)\\
\leq\ & (\ell-k)^2\frac{1}{2}\beta^2(\log c_n+M)-\big((\delta\ell k)^{1/2}\beta|x-y|/c_n\big)^2.
\end{aligned}$$

We thus obtain

$$I_n^1 \lesssim c_n^{-\beta^2}(c_nM')^{\frac{\beta^2}{2}(\ell-k)^2}\int_K\int_{\mathbf{R}^2} e^{-((\delta\ell k)^{1/2}\beta|x-y|/c_n)^2}\,dx\,dy \lesssim \frac{(c_nM')^{-\beta^2+2+\frac{\beta^2}{2}(\ell-k)^2}}{\ell k},$$

and this is transformed to the desired form as before.

Finally, the case where k and ℓ have different sign is much easier since then the term $\ell k\beta^2 C_n(x,y)$ has negative sign and works to our favour. □

We are now in a position to prove our universality result.

Theorem 3.18. (i) *Let $(X_n)_{n\geq 1}$ be a convolution approximation of a log-correlated field X as in Lemma 3.17 and let $0<\beta<\sqrt{d}$. Assume that $H:\mathbf{R}\to\mathbf{R}$ is a $2\pi/\beta$-periodic even function with absolutely convergent Fourier series and mean zero. Then there is a constant a such that for every test function $\varphi\in C_c^\infty(U)$ we have*

$$\int_U \varphi(x)e^{\frac12\beta^2C_n(x,x)}H(X_n(x))dx \to \langle a\,\text{“}\cos(\beta X)\text{”},\varphi\rangle,$$

in probability as $n\to\infty$.

(ii) *If $d=2$ and X_n, X satisfy the condition of part* (ii) *of the previous lemma, we have the same conclusion as in part* (i) *of this theorem, but assuming only that H is a locally integrable $2\pi/\beta$-periodic even function with mean zero.*

Proof. (i) Let $H(x)=\sum_{k=1}^\infty \widehat{H}_k\cos(\beta kx)$. By Theorem 1.1 it is enough to check that for a test function φ the quantity

$$R := \sum_{|k|\geq 2}\widehat{H}_k\int_U e^{\frac12\beta^2C_n(x,x)}\varphi(x)\big(e^{ik\beta X_n(x)}+e^{-ik\beta X_n(x)}\big)dx$$

converges to zero in probability. Since $\sum_{|k|\geq 2}|\widehat{H}_k|<\infty$ by assumption, this follows from Lemma 3.17(i) combined with two basic Cauchy–Schwarz estimates as one then finds that $\mathbb{E}\,R^2\to 0$ as $n\to\infty$.

(ii) We aim to show that again $\mathbb{E}\,R'^2\to 0$ as $n\to\infty$, where we now define

$$R' := \sum_{|k|\geq k_0}\widehat{H}_k\int_U e^{\frac12\beta^2C_n(x,x)}\varphi(x)\big(e^{ik\beta X_n(x)}+e^{-ik\beta X_n(x)}\big)dx,$$

where $k_0>2\sqrt{d}/\beta$. The finite number of terms with $2\leq|k|<k_0$ can be handled as in case (i). Since e.g. Fejér partial sums of the Fourier series converge to H almost everywhere pointwise, Fatou's lemma allows us to assume that H is a trigonometric polynomial and it is enough to prove a uniform bound for $\mathbb{E}\,R^2$ over all trigonometric polynomials H such that the modulus of all of their Fourier coefficients is bounded by 1. However, by Lemma 3.17(ii) we obtain in this situation

$$\begin{aligned}\mathbb{E}\,R'^2 &\leq \sum_{|k|,|\ell|\geq k_0}|\widehat{H}(k)\overline{\widehat{H}(\ell)}\mathbb{E}\,Y_k\overline{Y_\ell}| \lesssim c_n^{-\beta^2+2}\sum_{|k|,|\ell|\geq k_0}\frac{(c_ne^{2M})^{\frac{\beta^2}{4}(\ell-k)^2}}{(|\ell|\vee|k|)^2}\\ &\lesssim c_n^{-\beta^2+2}\to 0 \quad\text{as}\quad n\to\infty.\end{aligned}$$

□

Remark 3.19. The second part of the result applies to e.g. $*$-scale invariant log-correlated fields since they typically have translation invariant covariance structure. The same proof of course yields that if H is any complex valued $2\pi/\beta$-periodic function with zero mean and absolutely convergent Fourier series, the limit is a linear combination of the imaginary chaoses “$e^{\pm i\beta X}$”.

This concludes our study of universality and now we discuss the behavior of μ near β_c.

3.5. **Approach to the critical point.** As we have mentioned before and as follows from results in [57], $e^{\frac{\beta^2}{2}\mathbb{E}[X_n(x)^2]+i\beta X_n(x)}$ does not converge for $\beta \geq \sqrt{d}$, at least if one assumes a bit more of g and the approximation X_n. Nevertheless, if one multiplies this quantity by a suitable deterministic one, then one can prove convergence to white noise. In this section, we study how this fact that $\beta_c := \sqrt{d}$ is a special point can be seen from the limiting objects μ. In what follows, we find it convenient to write μ_β to indicate the dependence on β and hope this notation causes no confusion. The main result of this section is the following which describes how μ_β blows up as β increases to $\sqrt{d}$. The theorem complements in a natural manner some results in [57], and the methods used in the proof are somewhat similar to the ones already employed in that paper.

Theorem 3.20. *Let X be a log correlated field on the bounded subdomain $U \subset \mathbf{R}^d$ satisfying the standard assumptions* (2.2) *as before. Fix any test function $f \in C_c^\infty(U)$. As $\beta \nearrow \sqrt{d}$, we have*

$$\sqrt{\frac{d-\beta^2}{|S^{d-1}|}}\mu_\beta(f) \to \int_U f(x)e^{\frac{\beta^2}{2}g(x,x)}W(dx)$$

in law, where W is the standard complex white noise on U,[19] *and $|S^{d-1}|$ denotes the "area" of the unit sphere of $\mathbf{R}^d$.*

Proof. As we are dealing with Gaussian random variables, it is enough to show that the moments converge and we start by computing the second absolute one; we will implicitly be using constantly the results from Section 3.2 which allow us to write all the moments as suitable integrals. We have

$$\begin{aligned}\frac{d-\beta^2}{|S^{d-1}|}\mathbb{E}\,|\mu_\beta(f)|^2 &= \frac{d-\beta^2}{|S^{d-1}|}\int_{|x-y|<(d-\beta^2)^{\frac{1}{2d}}} f(x)\overline{f(y)}\frac{e^{\beta^2 g(x,y)}}{|x-y|^{\beta^2}}\,dx\,dy \\ &\quad + \frac{d-\beta^2}{|S^{d-1}|}\int_{|x-y|>(d-\beta^2)^{\frac{1}{2d}}} f(x)\overline{f(y)}\frac{e^{\beta^2 g(x,y)}}{|x-y|^{\beta^2}}\,dx\,dy.\end{aligned}$$

The trivial estimate $\frac{1}{|x-y|^{\beta^2}} \leq \frac{1}{(d-\beta^2)^{\frac{\beta^2}{2d}}}$ shows that the second term goes to 0 as $\beta \nearrow \sqrt{d}$. This and uniform continuity of our test function f and the function g on the support of f easily gives us

$$\begin{aligned}\lim_{\beta\nearrow\sqrt{d}}\frac{d-\beta^2}{|S^{d-1}|}\mathbb{E}\,|\mu_\beta(f)|^2 &= \lim_{\beta\nearrow\sqrt{d}}\frac{d-\beta^2}{|S^{d-1}|}\int_{|x-y|<(d-\beta^2)^{\frac{1}{2d}}}\frac{|f(x)|^2 e^{\beta^2 g(x,x)}}{|x-y|^{\beta^2}}\,dx\,dy \\ &= \lim_{\beta\nearrow\sqrt{d}}\frac{d-\beta^2}{|S^{d-1}|}\int_U |f(x)|^2 e^{\beta^2 g(x,x)}\int_{y\in B(x,(d-\beta^2)^{\frac{1}{2d}})}|x-y|^{-\beta^2}\,dy\,dx \\ &= \lim_{\beta\nearrow\sqrt{d}}\frac{d-\beta^2}{|S^{d-1}|}\int_U |f(x)|^2 e^{\beta^2 g(x,x)}|S^{d-1}|\int_0^{(d-\beta^2)^{\frac{1}{2d}}} r^{d-1-\beta^2}\,dr\,dx \\ &= \lim_{\beta\nearrow\sqrt{d}}(d-\beta^2)^{\frac{d-\beta^2}{2d}}\int_U |f(x)|^2 e^{\beta^2 g(x,x)}\,dx \\ &= \int_U |f(x)|^2 e^{\beta^2 g(x,x)}\,dx.\end{aligned}$$

Next note that for mixed moments we have by Lemma A.2 and the above computation that

$$\begin{aligned}\Big(\frac{d-\beta^2}{|S^{d-1}|}\Big)^{\frac{a+b}{2}}\Big|\mathbb{E}\,\mu_\beta(f)^a\overline{\mu_\beta(f)}^b\Big| &\leq C_{a,b}\Big(\frac{d-\beta^2}{|S^{d-1}|}\Big)^{\frac{a+b}{2}}(\mathbb{E}\,|\mu_\beta(f)|^2)^{\min(a,b)} \\ &\lesssim (d-\beta^2)^{\frac{a+b}{2}-\min(a,b)},\end{aligned}$$

[19] Our notation here is slightly formal; $Z_h := \int h(x)W(dx)$ denotes a centered complex Gaussian random variable satisfying $\mathbb{E}\,Z_h^2 = 0$ and $\mathbb{E}\,|Z_h|^2 = \int_U |h(x)|^2 dx$.

where the right hand side tends to 0 as $\beta \nearrow \sqrt{d}$. Thus it remains to check that the moments $\left(\frac{d-\beta^2}{|S^{d-1}|}\right)^a \mathbb{E}\,|\mu_\beta(f)|^{2a}$ behave correctly. We have

$$\mathbb{E}\,|\mu_\beta(f)|^{2a} = \int_{U^{2a}} \Big(\prod_{j=1}^a dx_j dy_j f(x_j)\overline{f(y_j)}\Big) \frac{\prod_{1\le j<k\le a} |x_j-x_k|^{\beta^2}|y_j-y_k|^{\beta^2} e^{-\beta^2 g(x_j,x_k)-\beta^2 g(y_j,y_k)}}{\prod_{1\le j,k\le a}|x_j-y_k|^{\beta^2} e^{-\beta^2 g(x_j,y_k)}}.$$

We may split the integration domain into the $a!$ disjoint sets A_σ, $\sigma\in S_a$, and the complement of their union, where

$$\begin{aligned} A_\sigma = \{|x_i - y_{\sigma_i}| < (d-\beta^2)^{\frac{1}{2d}} \text{ for all } 1\le i\le a\} \\ \cap\, \{|x_i-x_j| > (d-\beta^2)^{\frac{1}{3d}} \text{ for all } 1\le i<j\le a\}. \end{aligned}$$

Consider the integral over A_e, where e is the identity permutation. In A_e we have for $j<k$ that

$$\frac{|x_j-x_k|}{|x_j-y_k|} \ge \frac{|x_j-x_k|}{|x_j-x_k|+|x_k-y_k|} \ge \frac{1}{1+\frac{(d-\beta^2)^{\frac{1}{2d}}}{(d-\beta^2)^{\frac{1}{3d}}}}$$

and

$$\frac{|x_j-x_k|}{|x_j-y_k|} \le \frac{|x_j-x_k|}{|x_j-x_k|-|x_k-y_k|} \le \frac{1}{1-\frac{(d-\beta^2)^{\frac{1}{2d}}}{(d-\beta^2)^{\frac{1}{3d}}}}$$

from which we deduce that in A_e

$$\frac{|x_j-x_k|}{|x_j-y_k|} \to 1$$

as $\beta\to\sqrt{d}$. Similar reasoning shows that

$$\frac{|y_j-y_k|}{|x_k-y_j|} \to 1.$$

Hence again by uniform continuity of g and f

$$\begin{aligned} &\lim_{\beta\nearrow\sqrt{d}} \Big(\frac{d-\beta^2}{|S^{d-1}|}\Big)^a \int_{A_e} \Big(\prod_{j=1}^a dx_j dy_j f(x_j)\overline{f(y_j)}\Big) \frac{\prod_{1\le j<k\le a} |x_j-x_k|^{\beta^2}|y_j-y_k|^{\beta^2} e^{-\beta^2 g(x_j,x_k)-\beta^2 g(y_j,y_k)}}{\prod_{1\le j,k\le a}|x_j-y_k|^{\beta^2} e^{-\beta^2 g(x_j,y_k)}} \\ &= \lim_{\beta\nearrow\sqrt{d}} \Big(\frac{d-\beta^2}{|S^{d-1}|}\Big)^a \int_{A_e} \frac{\prod_{j=1}^a dx_j dy_j |f(x_j)|^2 e^{\beta^2 g(x_j,x_j)}}{\prod_{1\le j\le a}|x_j-y_j|^{\beta^2}} \\ &= \lim_{\beta\nearrow\sqrt{d}} \Big(\frac{d-\beta^2}{|S^{d-1}|}\Big)^a \int_{|x_i-x_j|>(d-\beta^2)^{\frac{1}{3d}}} \Big(\prod_{j=1}^a dx_j |f(x_j)|^2 e^{\beta^2 g(x_j,x_j)}\Big) \prod_{j=1}^a \int_{|y_j-x_j|<(d-\beta^2)^{\frac{1}{2d}}} \frac{dy_j}{|x_j-y_j|^{\beta^2}} \\ &= \lim_{\beta\nearrow\sqrt{d}} \Big(\frac{d-\beta^2}{|S^{d-1}|}\Big)^a \int_{|x_i-x_j|>(d-\beta^2)^{\frac{1}{3d}}} \Big(\prod_{j=1}^a dx_j |f(x_j)|^2 e^{\beta^2 g(x_j,x_j)}\Big) |S^{d-1}|^a \Big(\int_0^{(d-\beta^2)^{\frac{1}{2d}}} r^{d-1-\beta^2}\,dr\Big)^a \\ &= \Big(\int_U |f(x)|^2 e^{\beta^2 g(x,x)}\,dx\Big)^a. \end{aligned}$$

By relabelling y_i, we see that the result does not depend on the permutation chosen, so we get the same outcome $a!$ times. Thus the moments converge to Gaussian ones as soon as we check that the contribution from the complement of the sets A_σ goes to 0. The complement is covered by the sets

$$B_1 = \{|x_j-x_k| \le (d-\beta^2)^{\frac{1}{3d}} \text{ for some } 1\le j<k\le a\}$$

and

$$B_{2,k} = \{|x_k-y_j| > (d-\beta^2)^{\frac{1}{2d}} \text{ for all } j\ne k\}.$$

We have

$$\lim_{\beta\to\sqrt{d}}\Big(\frac{d-\beta^2}{|S^{d-1}|}\Big)^a\int_{B_1}\Big(\prod_{j=1}^a dx_jdy_jf(x_j)\overline{f(y_j)}\Big)\frac{\prod_{1\le j<k\le a}|x_j-x_k|^{\beta^2}|y_j-y_k|^{\beta^2}e^{-\beta^2g(x_j,x_k)-\beta^2g(y_j,y_k)}}{\prod_{1\le j,k\le a}|x_j-y_k|^{\beta^2}e^{-\beta^2g(x_j,y_k)}}$$
$$=0$$

because we may use Lemma A.1 and Fubini's theorem to integrate out the variables y_k, leaving a term of size $\lesssim (d-\beta^2)^{-a}$ that cancels the factor in front. The remaining integral over the variables x_k is over a domain whose measure goes to 0. Finally, again using Lemma A.1 we have

$$\Big(\frac{d-\beta^2}{|S^{d-1}|}\Big)^a\int_{B_{2,a}}\Big(\prod_{j=1}^a dx_jdy_jf(x_j)\overline{f(y_j)}\Big)\frac{\prod_{1\le j<k\le a}|x_j-x_k|^{\beta^2}|y_j-y_k|^{\beta^2}e^{-\beta^2g(x_j,x_k)-\beta^2g(y_j,y_k)}}{\prod_{1\le j,k\le a}|x_j-y_k|^{\beta^2}e^{-\beta^2g(x_j,y_k)}}$$
$$\lesssim \|f\|_\infty^{2a}\sum_{\sigma\in S_a}\Big(\frac{d-\beta^2}{|S^{d-1}|}\Big)^a\int_{B_{2,a}}\Big(\prod_{j=1}^a dx_jdy_j\Big)\frac{1}{\prod_{1\le j\le a}|x_j-y_{\sigma_j}|^{\beta^2}}$$
$$\lesssim \sum_{\sigma\in S_a}\Big(\frac{d-\beta^2}{|S^{d-1}|}\Big)^a(d-\beta^2)^{-\frac{\beta^2}{2d}}\int_{B_{2,a}}\Big(\prod_{j=1}^a dx_jdy_j\Big)\frac{1}{\prod_{1\le j\le a-1}|x_j-y_{\sigma_j}|^{\beta^2}}$$
$$\lesssim (d-\beta^2)^{1-\frac{\beta^2}{2d}},$$

which goes to 0. A similar calculation holds for $B_{2,k}$, $1\le k\le a-1$. □

This concludes the portion of this article dealing with basic properties of imaginary chaos. We now turn to discussing the Ising model.

4. The Ising model and multiplicative chaos: the scaling limit of the critical and near critical planar XOR-Ising spin field

The goal of this section is to prove Theorem 1.5 and Theorem 1.6. We begin by first recalling the definition of the Ising model (with + boundary conditions) on a finite part of the square lattice as well as recent results concerning the scaling limit of correlation functions of the spin field for the critical Ising model on a finite part of the square lattice. We then define the XOR-Ising model on the square lattice and using the results concerning the correlation functions (along with some rough estimates for the behavior of the correlation functions on the diagonals), we prove Theorem 1.5, namely that in zero magnetic field, the scaling limit of the critical XOR-Ising spin field is the real part of an imaginary multiplicative chaos distribution. After this, we prove that if we add a magnetic field to the XOR-Ising model, then the scaling limit of the spin field can be seen as the cosine of the sine-Gordon field, which is Theorem 1.6.

4.1. The Ising model and spin correlation functions for the critical planar Ising model.

Let $U\subset \mathbf{C}$ be a simply connected bounded planar domain, and for $\delta>0$, let F_δ be the set of faces of the lattice graph $\delta\mathbf{Z}^2$ that are contained in U. To avoid overlap, let us say that the faces are half-open, i.e. of the form $\delta([n,n+1)\times[m,m+1))$ for some $m,n\in\mathbf{Z}$. Following [18], we will define our Ising model on the faces F_δ. We also define the set of boundary faces ∂F_δ as the set of those faces in $\delta\mathbf{Z}^2$ which are adjacent to a face in F_δ but not in F_δ themselves.

We call a function $\sigma\colon F_\delta\cup\partial F_\delta\to\{-1,1\}$, $a\mapsto\sigma_a$ a spin configuration on $F_\delta\cup\partial F_\delta$ and we define the Ising model on F_δ with + boundary conditions, inverse temperature β, and zero magnetic field to be a probability measure on the set of spin configurations on $F_\delta\cup\partial F_\delta$ such that the law of the spin configuration is

$$\mathbb{P}_\delta(\sigma)=\mathbb{P}^+_{\delta,\beta,U}(\sigma)=\frac{1}{Z_\beta}e^{\beta\sum_{a,b\in F_\delta\cup\partial F_\delta,a\sim b}\sigma_a\sigma_b}\mathbf{1}\{\sigma_{|\partial F_\delta}=1\},$$

where by $a\sim b$ we mean that $a,b\in F_\delta\cup\partial F_\delta$ are neighboring faces, and Z_β is a normalizing constant. We count each pair a,b of nearest neighbor faces only once. We will want to talk about the spin at an arbitrary point $x\in U$, so we define a function $\sigma_\delta(x)=\sigma_f$ if $x\in f\in F_\delta$, and $\sigma_\delta(x)=1$ otherwise.

As discussed in the introduction, a fundamental fact about the planar Ising model with zero magnetic field is that the model has a phase transition. From now on, we will focus on the critical model, namely when $\beta = \beta_c = \frac{\log(1+\sqrt{2})}{2}$ – see [8, Section 7.12]. We will also write from now on $\mathbb{P}_\delta = \mathbb{P}^+_{\delta,\beta_c,U}$ for the law of the critical Ising model (on the faces $F_\delta \cup \partial F_\delta$ with the + boundary conditions as indicated above) as well as the law of the induced spin field $\sigma_\delta\colon U \to \{-1,1\}$.

We next turn to the analysis of the correlation functions of σ_δ, which as we discussed in Section 1.3 have a non-trivial scaling limit and are connected to conformal field theory. The precise statement concerning the scaling limit is a recent result of Chelkak, Hongler, and Izyurov (see [18, Theorem 1.2] and the discussion leading to it):

Theorem 4.1 (Chelkak, Hongler, and Izyurov)**.** *Let $x_1, ..., x_n \in U$ be distinct and the spin field σ_δ be distributed according to $\mathbb{P}_\delta$ (as defined above). Then for $\mathcal{C} = 2^{5/48} e^{\frac{3}{2}\zeta'(-1)}$,*

$$\lim_{\delta\to 0^+} \delta^{-\frac{n}{8}} \mathbb{E}\left[\prod_{j=1}^n \sigma_\delta(x_j)\right] = \mathcal{C}^n \prod_{j=1}^n \left(\frac{|\varphi'(x_j)|}{2\mathrm{Im}\varphi(x_j)}\right)^{1/8}$$
$$\times\left(2^{-n/2}\sum_{\mu\in\{-1,1\}^n}\prod_{1\le k<m\le n}\left|\frac{\varphi(x_k)-\varphi(x_m)}{\varphi(x_k)-\overline{\varphi(x_m)}}\right|^{\frac{\mu_k\mu_m}{2}}\right)^{1/2},$$

where $\varphi : U \to \mathbf{H} = \{x+iy \in \mathbf{C} : y > 0\}$ is any conformal bijection and for any $\varepsilon > 0$, the convergence is uniform in $\{x_1, ..., x_n \in \Omega : \min_{i\neq j} |x_i - x_j| > \varepsilon, \min_i d(x_i, \partial U) > \varepsilon\}$.

Remark 4.2. We note that in [18], the authors consider actually the square lattice rotated by $\pi/4$ and with diagonal mesh 2δ in which case the lattice spacing is $\sqrt{2}\delta$ instead of δ as in our case. Rotating the lattice plays a role only in the value of the constant $\mathcal{C}$. Our version follows by replacing their δ with $\delta/\sqrt{2}$. We also note that in [18] there appears to be a sign error in the exponent of $e^{\frac{3}{2}\zeta'(-1)}$. We offer here a brief suggestion on how the interested reader might convince themselves of this fact. First of all, as pointed out in [18, Remark 1.4], one can recover the (continuum) whole plane spin-correlation functions from the finite volume ones through a suitable limiting process. In particular, the scaling limit of the whole plane two-point function equals $\mathcal{C}^2|x-y|^{-1/4}$ (see [18, (1.6)]). On the other hand, it is known that on the whole plane $\mathbf{Z}^2$-lattice the *diagonal* two point function has an explicit product representation – see e.g. [64, (XI.4.18)]. This product can be written in terms of Barnes G functions, and using their known asymptotics, one can recover the correct value of $\mathcal{C}$. We thank Antti Suominen for pointing this sign error out to us. ■

4.2. **The critical XOR-Ising model and its magnetic perturbation.** Following Wilson [83], see also [13], we consider now the so called XOR-Ising model, which is again a probability measure on spin configurations, but now the spin configurations are given by a pointwise product of two independent Ising spin configurations. We focus on the critical case again and we thus make the following definitions: let $\sigma_\delta, \widetilde{\sigma}_\delta$ be independent and distributed according to $\mathbb{P}_\delta$ and define for $x \in U$, $\mathcal{S}_\delta(x) = \sigma_\delta(x)\widetilde{\sigma}_\delta(x)$. Also write for $a \in F_\delta$, $\mathcal{S}_a = \sigma_a\widetilde{\sigma}_a$. Let us write $\mathcal{P}_\delta$ for the law of $\mathcal{S}$ (both the spin configuration and spin field, and as for the normal Ising model, we don't care what space of functions $\mathcal{S}$ lives on). Perhaps slightly artificially, but as discussed in Section 1.3, motivated by wanting to study scaling limits of near critical models of statistical mechanics, we also add a coupling to a (non-uniform) magnetic field to this law: for a function $\psi \in C_c^\infty(U)$, define

$$\begin{aligned}\mathcal{P}_{\psi,\delta}(\mathcal{S}) &= \frac{1}{\mathcal{Z}_{\psi,\delta}} e^{\delta^{2-\frac{1}{4}}\sum_{a\in F_\delta\cup\partial F_\delta}(\delta^{-2}\int_a \psi(x)dx)\mathcal{S}_a}\mathcal{P}_\delta(\mathcal{S})\\ &= \frac{1}{\mathcal{Z}_{\psi,\delta}} e^{\delta^{-1/4}\int_U \psi(x)\mathcal{S}_\delta(x)dx}\mathcal{P}_\delta(\mathcal{S}),\end{aligned}$$

where $\mathcal{Z}_{\psi,\delta}$ is a normalizing constant. The reason to view this as a coupling to a magnetic field is that typically in spin models of statistical mechanics, the part of the energy of a spin configuration (σ_a) coming from an interaction with a magnetic field (h_a) is given by $-\sum_a h_a\sigma_a$, and in the Gibbs measure of the model in a non-zero magnetic field is obtained by biasing the zero-magnetic field Gibbs measure with a quantity $\frac{1}{Z_{\beta,h}}e^{\beta\sum_a h_a\sigma_a}$, where $Z_{\beta,h}$ is a normalizing constant. In this picture, our model corresponds roughly to choosing $h_a = \delta^{2-\frac{1}{4}}\psi(a)$ (where $\psi(a)$ means the value at the center of the face, which is close to $\delta^{-2}\int_a \psi(x)dx$ due to the smoothness of ψ). Since $h_a \to 0$ as $\delta \to 0$, one sometimes calls this type of model near-critical in that it is close to the critical case of $h = 0$.

4.3. **Convergence to multiplicative chaos.** The goal of this subsection is to prove Theorem 1.5. The main point in the proof is to obtain a δ-independent integrable upper bound for the n-point correlation function $\mathbb{E}\,\sigma_\delta(x_1)\ldots\sigma_\delta(x_n)$, which makes it possible to use the dominated convergence theorem and Theorem 4.1 to find asymptotics of moments of $\delta^{-1/4}\int_U \mathcal{S}_\delta(x)f(x)dx$ and then using the method of moments, justified by Theorem 1.3, conclude the convergence. Such an upper bound is obtained by proving a variant of Onsager's inequality for the Ising model, after which integrability is obtained again from Lemma 3.10.

The precise statement about the moments of $\mathcal{S}_\delta$ is the following.

Lemma 4.3. *For each $f \in C_c^\infty(U)$ and integer $k \geq 0$*

$$\lim_{\delta\to 0}\mathbb{E}\left(\delta^{-1/4}\int_U f(x)\mathcal{S}_\delta(x)dx\right)^k \tag{4.1}$$
$$= \left(\frac{\mathcal{C}^2}{\sqrt{2}}\right)^k \int_{U^k}\prod_{j=1}^k\left[f(x_j)\left(\frac{|\varphi'(x_j)|}{2\mathrm{Im}\,\varphi(x_j)}\right)^{1/4}\right]\sum_{\mu\in\{-1,1\}^n}\prod_{i<j}\left|\frac{\varphi(x_i)-\varphi(x_j)}{\varphi(x_i)-\overline{\varphi(x_j)}}\right|^{\frac{\mu_i\mu_j}{2}}\prod_{j=1}^k dx_j.$$

and for each $\lambda > 0$

$$\sup_{\delta>0}\mathbb{E}\, e^{\lambda\left|\delta^{-1/4}\int_U \mathcal{S}_\delta(x)dx\right|} < \infty. \tag{4.2}$$

Our proof will be based on the following lemma.

Lemma 4.4. *Let $a_1,\ldots,a_k \in F_\delta$ be distinct faces lying inside a fixed compact set $K \subset U$ and identify each face with its center. Then for some constant $C > 0$ we have*

$$\delta^{-k/8}\mathbb{E}\,\sigma_{a_1}\ldots\sigma_{a_k} \leq C^k\prod_{i=1}^k\left(\min_{j\neq i}|a_i-a_j|\right)^{-1/8}.$$

The constant C is independent of the points a_i, k and δ, but it may depend on K.

Proof. This inequality essentially appears in the proof of Proposition 3.10 in [38], where the authors show ([38, last line on p. 20]) that

$$\mathbb{E}\,\sigma_{a_1}\ldots\sigma_{a_k} \leq \prod_{i=1}^p \phi^+_{B_i}(a_i \leftrightarrow \partial B_i).$$

Here $B_i = a_i + [-\ell_i/4, \ell_i/4]^2$ are disjoint boxes with $\ell_i = \min_{j\geq 0, j\neq i} d(a_i, a_j)$ being the $\delta\mathbf{Z}^2$-distance (we have added the factor δ compared to [38] because we are working on the scaled lattice) from a_i to its closest neighbour or to the boundary ∂U which is denoted by a_0. The quantity $\phi^+_{B_i}(a_i \leftrightarrow \partial B_i)$ denotes the probability that a_i is connected to the boundary of B_i in the FK–Ising model (see e.g. [38, Section 3.1 and Section 3.2] and references therein), and this probability is less than $C\ell_i^{-1/8}$ by [38, Lemma 3.9]. Our claim then follows from the elementary inequality $d(a,b) \leq \sqrt{2}|a-b|/\delta$ and the fact that by compactness $d(K, \partial U)$ is bounded from below for small enough δ. $\square$

This allows us to give the proof of Lemma 4.3.

Proof of Lemma 4.3. Let $K \subset U$ be a fixed compact set and let $x_1,\ldots,x_k \in K$. We claim that for some $C > 0$ independent of x_i, k, and δ,

$$\delta^{-k/8}\mathbb{E}\,\sigma_\delta(x_1)\cdots\sigma_\delta(x_k)\leq C^k\prod_{i=1}^k\left(\min_{j\neq i}|x_i-x_j|\right)^{-1/8} \tag{4.3}$$

which can be seen as a variant of Onsager's inequality for the Ising model.

Let us write $a_i^{(x)}$ for the face x lies in (with the convention that we count in it the southern and western boundary without corners as well as the south-western corner). If we first assume that all of the $a_i^{(x)}$ are distinct ($|a_i^{(x)}-a_j^{(x)}|\geq\delta$) and note that $|x_i-x_j|\leq|x_i-a_i^{(x)}|+|a_i^{(x)}-a_j^{(x)}|+|x_j-a_j^{(x)}|\leq 2\delta+|a_i^{(x)}-a_j^{(x)}|\leq 3|a_i^{(x)}-a_j^{(x)}|$, then (4.3) follows immediately from Lemma 4.4.

Consider then the case where not all of the $a_i^{(x)}$ are distinct. After using $\sigma_a^2=1$ to reduce the number of spins from the correlation function and possibly relabelling the spins, let us assume that we have

$$\delta^{-k/8}\mathbb{E}\,\sigma_\delta(x_1)\cdots\sigma_\delta(x_k)=\delta^{-k/8}\mathbb{E}\,\sigma_\delta(x_1)\cdots\sigma_\delta(x_l)$$

with $l<k$ and $(a_i^{(x)})_{i=1}^l$ distinct. From the case where all faces were distinct, we find

$$\begin{aligned}\delta^{-k/8}\mathbb{E}\,\sigma_\delta(x_1)\cdots\sigma_\delta(x_k)&\leq\delta^{-(k-l)/8}C^l\prod_{j=1}^l\left(\min_{1\leq i\leq l,i\neq j}|x_i-x_j|\right)^{-1/8}\\&\leq\delta^{-(k-l)/8}C^l\prod_{j=1}^l\left(\min_{1\leq i\leq k,i\neq j}|x_i-x_j|\right)^{-1/8}\end{aligned}$$

where the second step comes from the fact that we minimize over a larger set. Now for the remaining points $x_{l+1},...,x_k$, for each of them, there is another x_i such that both points belong to the same face, implying that for $j>l$,

$$\min_{1\leq i\leq k,i\neq j}|x_i-x_j|\leq\sqrt{2}\delta$$

so that

$$\delta^{-(k-l)/8}\leq 2^{\frac{k-l}{16}}\prod_{j=l+1}^k\left(\min_{1\leq i\leq k,i\neq j}|x_i-x_j|\right)^{-1/8},$$

which concludes the proof of (4.3).

We may now compute

$$\begin{aligned}\lim_{\delta\to0}\mathbb{E}\left(\delta^{-1/4}\int_U f(x)\mathcal{S}_\delta(x)\,dx\right)^k&=\delta^{-k/4}\int_{U^k}f(x_1)\dots f(x_k)\mathbb{E}\,\mathcal{S}_\delta(x_1)\dots\mathcal{S}_\delta(x_k)\,dx\\&=\delta^{-k/4}\int_{U^k}f(x_1)\dots f(x_k)(\mathbb{E}\,\sigma_\delta(x_1)\dots\sigma_\delta(x_k))^2\,dx.\end{aligned}$$

Using (4.3), we see that the absolute value of the integrand is at most

$$C^{2k}\|f\|_\infty^k\prod_{i=1}^k\left(\min_{j\neq i}|x_i-x_j|\right)^{-1/4}.$$

By Lemma 3.10 this is integrable, so we may apply the dominated convergence theorem and Theorem 4.1 to get

$$\begin{aligned}
&\lim_{\delta\to 0}\mathbb{E}\left(\delta^{-1/4}\int_U f(x)\mathcal{S}_\delta(x)\,dx\right)^k\\
&=\int_{U^k} f(x_1)\dots f(x_k)\mathcal{C}^{2k}\prod_{j=1}^k\left(\frac{|\varphi'(x_j)|}{2\operatorname{Im}\varphi(x_j)}\right)^{1/4}2^{-k/2}\sum_{\mu\in\{-1,1\}^n}\prod_{1\le i<j\le k}\left|\frac{\varphi(x_i)-\varphi(x_j)}{\varphi(x_i)-\overline{\varphi(x_j)}}\right|^{\frac{\mu_i\mu_j}{2}}\,dx,
\end{aligned}$$

which proves (4.1). Moreover, the uniform bound obtained from Lemma 3.10 also implies (4.2). □

Having Lemma 4.3 in our hand, we can now turn to the proof of convergence to chaos.

Proof of Theorem 1.5. By (the proof of) Theorem 1.3, the moments of

$$\int_U \mathcal{C}^2\left(\frac{2|\varphi'(x)|}{\operatorname{Im}\varphi(x)}\right)^{1/4}\cos(2^{-1/2}X(x))f(x)\,dx$$

are precisely the right side of (4.1). Thus by Lemma 4.3 the moments of the XOR-Ising field converge to those of the real part of the imaginary chaos and by Theorem 1.3, the moments of the imaginary chaos grow slowly enough so that they determine its distribution and the convergence of moments implies convergence in law – see Corollary 3.13. □

4.4. **The sine-Gordon model.** Let us now introduce the sine-Gordon type model appearing in the statement of Theorem 1.6. In the theoretical physics literature, a definition of the sine-Gordon model could be representing the correlation functions of the sine-Gordon field as a functional integral, which might be written as

$$\langle X(x_1)\cdots X(x_k)\rangle_{\mathrm{sG}(\lambda,\beta)}=\frac{1}{Z(\lambda,\beta)}\int X(x_1)\cdots X(x_k)e^{\lambda\int_{\mathbf{R}^2}\cos\beta X(x)dx-\int_{\mathbf{R}^d}\nabla X(x)\cdot\nabla X(x)dx}\mathcal{D}X.$$

Above $\mathcal{D}X=\prod_{x\in\mathbf{R}^2}dX(x)$ is formally the (non-existent) infinite dimensional Lebesgue measure and the integral is over $\mathbf{R}^{\mathbf{R}^2}$. This is of course ill-defined, but the way one mathematically makes sense of this is through understanding the combination $e^{-\int_{\mathbf{R}^2}\nabla X(x)\cdot\nabla X(x)dx}\mathcal{D}X$ as the probability distribution of the (whole plane) Gaussian free field. Then one could try to view this as biasing the law of the Gaussian free field with something again related to imaginary multiplicative chaos. For our purposes, it is more convenient to work in a finite domain with zero boundary conditions on the free field (this also avoids the problem with the zero mode or the fact that the whole plane free field is well defined only up to a random additive constant). Also instead of having just the quantity $\lambda\int\cos\beta X(x)dx$, our purposes require generalizing slightly and replacing the constant λ by a weight in the integral. We thus make the following definition.

Definition 4.5. Let $U\subset\mathbf{R}^2$ be a bounded simply connected domain, let X be the zero boundary Gaussian free field in U – see Example 2.6 – with law $\mathbb{P}_{\mathrm{GFF}}$ on (say) $H^{-\varepsilon}(\mathbf{R}^2)$[20]. For $\psi\in C_c^\infty(U)$, $\beta\in(0,\sqrt{2})$, the sine-Gordon(ψ,β) model in domain U with zero boundary condition is a probability distribution on $H^{-\varepsilon}(\mathbf{R}^2)$ of the form

$$\mathbb{P}_{\mathrm{sG}(\psi,\beta)}(dX)=\frac{1}{Z(\psi,\beta)}e^{\int_U\psi(x)\cos\beta X(x)dx}\mathbb{P}_{\mathrm{GFF}}(dX),$$

where again the integral in the exponential is formal notation for testing the random generalized function $\cos(\beta X)$ against the test function ψ. ■

Remark 4.6. For the above definition to make sense, $\cos\beta X$ has to be measurable w.r.t. X and we need $\mathbb{E}\,e^{\int_U\psi(x)\cos\beta X(x)\,dx}$ to be finite. The first property follows simply from our convergence in probability in Theorem 1.1, while the second one follows from Theorem 1.3. ■

[20]Recall that the field X is actually supported in U – see Proposition 2.3. As is often done, one could also consider X as a random element of $H^{-\varepsilon}(U)$.

This definition allows us to construct the cosine of the sine-Gordon field, namely the proposed limiting object from Theorem 1.6. We do not really need to construct it as a random generalized function, we simply need to know that for each test function, there exists a random variable that can be viewed as the cosine of the sine-Gordon field tested against this test function.

Definition 4.7. Let $U \subset \mathbf{R}^2$ be a bounded simply connected domain. For each $\beta, \gamma \in (0, \sqrt{2})$ and $f, \psi \in C_c^\infty(U)$, let us write

$$\int_U f(x) \cos\left(\gamma X_{\mathrm{sG}(\psi,\beta)}(x)\right) dx$$

for the random variable whose law is characterized by the condition that for each bounded continuous $F : \mathbf{R} \to \mathbf{R}$,

$$\begin{aligned}\mathbb{E}&\left[F\left(\int_U f(x)\cos\left(\gamma X_{\mathrm{sG}(\psi,\beta)}(x)\right) dx\right)\right] \\ &= \frac{1}{Z(\psi,\beta)}\mathbb{E}_{\mathrm{GFF}}\left[F\left(\int_U f(x)\cos\left(\gamma X(x)\right) dx\right) e^{\int_U \psi(x)\cos(\beta X(x))dx}\right],\end{aligned}$$

where $\int_U f(x)\cos(\gamma X(x))dx$ and $\int_U \psi(x)\cos(\beta X(x))dx$ denote the action of the real parts of imaginary chaos distributions built from the GFF on U with zero boundary conditions provided by Theorem 1.1. ■

To see that this is a valid definition, first note from Theorem 1.1 that we can simultaneously construct both of the random variables $\int_U f(x)\cos(\gamma X(x))dx$ and $\int_U \psi(x)\cos(\beta X(x))dx$ on the same probability space. Moreover, as F is bounded, we have from Theorem 1.3 that the expectation on the right hand side of the equation in the definition is finite. Thus, by the standard argument of interpreting this as a positive linear functional of F, the Riesz–Markov–Kakutani representation theorem provides the existence of the desired probability distribution. We note that one could also construct the same object starting from regularizations of the free field.

We are now in a position to move on to the proof of Theorem 1.6.

4.5. **Convergence of the magnetically perturbed critical XOR-Ising to cosine of the sine-Gordon field.** Proving that the spin field of the magnetically perturbed XOR-Ising model converges to the cosine of the sine-Gordon field, or Theorem 1.6, now follows rather easily from Theorem 1.1.

Proof of Theorem 1.6. What we wish to show is that for each bounded continuous $F : \mathbf{R} \to \mathbf{R}$,

$$\begin{aligned}\lim_{\delta\to 0}&\mathbb{E}_{\psi,\delta}\left[F\left(\delta^{-1/4}\int_U f(x)\mathcal{S}_\delta(x)dx\right)\right] \\ &= \mathbb{E}\left[F\left(\mathcal{C}^2\int_U \left(\frac{2|\varphi'(x)|}{\operatorname{Im}\varphi(x)}\right)^{1/4} f(x)\cos\left(2^{-1/2}X_{\mathrm{sG}(\widetilde{\psi},1/\sqrt{2})}(x)\right) dx\right)\right],\end{aligned}$$

where on the left hand side we have the spin field of the magnetically perturbed XOR-Ising model, with law $\mathcal{P}_{\psi,\delta}$ and expectation $\mathbb{E}_{\psi,\delta}$, and on the right hand side we have the random variable defined in Definition 4.7.

Recall that we wrote $\mathcal{P}_\delta$ for the law of the spin field of the zero-magnetic field XOR-Ising model and let us write $\mathbb{E}_\delta$ for the corresponding expectation. By the definition of $\mathcal{P}_{\psi,\delta}$ we thus have

$$\mathbb{E}_{\psi,\delta}F\Big(\delta^{-1/4}\int_U \mathcal{S}_\delta(x)f(x)\,dx\Big) = \frac{1}{\mathcal{Z}_{\psi,\delta}}\mathbb{E}_\delta\left[F\Big(\delta^{-1/4}\int_U \mathcal{S}_\delta(x)f(x)\,dx\Big)e^{\delta^{-1/4}\int_U \psi(x)\mathcal{S}_\delta(x)\,dx}\right]$$

By Theorem 1.5 we know that under $\mathcal{P}_\delta$, $\delta^{-1/4}\mathcal{S}_\delta$ tested against an arbitrary test function converges in law to $\cos(\frac{1}{\sqrt{2}}X)$ (where X is the free field) tested against $\mathcal{C}^2(2\frac{|\varphi'(x)|}{\operatorname{Im}\varphi(x)})^{1/4}$ times that same test function, so by linearity and the Cramér-Wold theorem, the random variables $A = \delta^{-1/4}\int_U \mathcal{S}_\delta(x)f(x)\,dx$ and $B = \delta^{-1/4}\int_U \mathcal{S}_\delta(x)\psi(x)\,dx$ converge jointly in law (to the corresponding random variables expressed in terms of the free field. From the continuity of the map $(x,y)\mapsto F(x)e^y$ it follows that also $F(A)e^B$ converges in law by the continuous mapping theorem [54, Lemma 4.27] to the random variable

$$F\left(\mathcal{C}^2\int_U\left(\frac{2|\varphi'(x)|}{\operatorname{Im}\varphi(x)}\right)^{1/4}f(x)\cos(2^{-1/2}X(x))dx\right)e^{\int_U\widetilde{\psi}(x)\cos(2^{-1/2}X(x))dx}.$$

Moreover, by the (exponential) uniform integrability provided by boundedness of exponential moments proven in Lemma 4.3, $\mathcal{Z}_{\psi,\delta}$ converges to $Z(\psi,1/\sqrt{2})$ as $\delta\to 0$. These remarks combined with another application of the boundedness of exponential moments from Lemma 4.3 shows that also the expectation of $F(A)e^B$ converges to the correct quantity as $\delta\to 0$ and we deduce that

$$\begin{aligned}&\lim_{\delta\to 0}\mathbb{E}_{\psi,\delta}\left[F\left(\delta^{-1/4}\int_U f(x)\mathcal{S}_\delta(x)dx\right)\right]\\&\quad=\mathbb{E}\left[F\left(\mathcal{C}^2\int_U\left(\frac{2|\varphi'(x)|}{\operatorname{Im}\varphi(x)}\right)^{1/4}f(x)\cos\left(2^{-1/2}X_{\mathrm{sG}(\widetilde{\psi},1/\sqrt{2})}(x)\right)dx\right)\right],\end{aligned}$$

as was desired. ☐

This concludes our study of the Ising model.

5. Random unitary matrices and imaginary multiplicative chaos

The goal of this section is to prove Theorem 1.7. As mentioned in Section 1.4, the proof is very similar to that in [81]. Below, we first recall our model and some known results about it, after which we record the proof which follows rather directly from the known results.

5.1. **Defining the fields.** As mentioned in Section 1.4, the random matrix model we consider is that of Haar distributed random unitary matrices. That is, we consider a random $N\times N$ unitary matrix U_N whose distribution is given by the Haar, or uniform measure on the unitary group $\mathrm{U}(N)$.[21]

As discussed in Section 1.4, we wish to consider objects related to the characteristic polynomial of U_N which give rise to imaginary multiplicative chaos. The relevant fields to consider are the real and imaginary parts of the logarithm of the characteristic polynomial evaluated on the unit circle. More precisely, we define for $\theta\in[0,2\pi]$

$$X_N(\theta)=\log|\det(I-e^{-i\theta}U_N)|\quad\text{and}\quad Y_N(\theta)=\lim_{r\to 1^-}\operatorname{Im}\operatorname{Tr}\log(I-re^{-i\theta}U_N),$$

where I denotes the $N\times N$ identity matrix, and in the definition of Y_N, what we mean by $\operatorname{Tr}\log(I-re^{-i\theta}U_N)$ is $\sum_{j=1}^N\log(1-re^{i(\theta_j-\theta)})$, where $(e^{i\theta_j})_{j=1}^N$ are the eigenvalues of U_N, and the branch of the logarithm is the principal one – namely it is given by $\log(1-z)=-\sum_{k=1}^\infty\frac{1}{k}z^k$ for $|z|<1$. Note that in this case, the limit defining Y_N exists almost surely e.g. in $L^2([0,2\pi],d\theta)$. Moreover, an elementary exercise in trigonometry shows that if $\theta\neq\theta_j$ for all j, then we can write

$$\begin{aligned}Y_N(\theta)&=\sum_{j=1}^N\frac{\theta_j-\theta}{2}+\frac{\pi}{2}\sum_{j=1}^N(\mathbf{1}\{\theta_j<\theta\}-\mathbf{1}\{\theta_j>\theta\})\\&=\sum_{j=1}^N\frac{\theta_j-\theta}{2}-\frac{N\pi}{2}+\pi\sum_{j=1}^N\mathbf{1}\{\theta_j<\theta\}\\&=-\frac{1}{2i}\sum_{k=1}^\infty\frac{1}{k}\left(e^{-ik\theta}\operatorname{Tr}U_N^k-e^{ik\theta}\operatorname{Tr}U_N^{-k}\right),\end{aligned}\tag{5.1}$$

[21]Being one of the classical compact groups, it is a classical fact that there exists a unique probability measure $\mathbb{P}_N$ on $\mathrm{U}(N)$ such that for any Borel set $\mathcal{B}\subset\mathrm{U}(N)$ and any fixed $U\in\mathrm{U}(N)$, $\mathbb{P}_N(U\mathcal{B})=\mathbb{P}_N(\mathcal{B}U)=\mathbb{P}_N(\mathcal{B})$ – this probability measure is the one we take for the distribution of the random matrix U_N. We write simply $\mathbb{E}$ for integration with respect to $\mathbb{P}_N$

where we have used the convention that $\theta_j \in [0, 2\pi)$ for all j, and $\theta \in [0, 2\pi)$. Moreover, the last equality is valid in the L^2-sense and is a consequence of the series expansion of the logarithm in the unit circle. We now review some standard facts relating random unitary matrices to Toeplitz determinants and recent results concerning asymptotics of such determinants.

5.2. **Random unitary matrices and asymptotics of Toeplitz determinants.** A fundamental fact about Haar distributed random unitary matrices is an exact formula for the distribution of the eigenvalues of the matrix. More precisely, a consequence of Weyl's integration formula is that the distribution of $(e^{i\theta_j})_{j=1}^N$ (for $\theta_j \in [0, 2\pi]$) is given by

$$\frac{1}{N!} \prod_{k<l} \left| e^{i\theta_k} - e^{i\theta_l} \right|^2 \prod_{j=1}^{k} \frac{d\theta_j}{2\pi}. \tag{5.2}$$

From this, using the fact that the product over pairs here can be expressed in terms of the Vandermonde determinant, one can check the so-called Heine–Szegő identity: if we write $\mathbb{T}$ for the unit circle of the complex plane, then for $h \in L^1(\mathbb{T}, \mathbf{C})$

$$\mathbb{E} \prod_{j=1}^{N} h\big(e^{i\theta_j}\big) = \det \left(\int_0^{2\pi} e^{-i(k-l)\theta} h\big(e^{i\theta}\big) \frac{d\theta}{2\pi} \right)_{k,l=0}^{N-1}. \tag{5.3}$$

As we will shortly see, we'll make use of this connection. In particular, what will be sufficient for our purposes is the fact that for say a smooth function $V : \mathbb{T} \to \mathbf{C}$ with no winding and real parameters $\beta_1, \beta_2, \gamma_1, \gamma_2$ as well as distinct points $\varphi, \varphi' \in [0, 2\pi]$, one finds from (5.1) as well as the Heine–Szegő identity (5.3):

$$\begin{aligned} &\mathbb{E}\, e^{\sum_{j=1}^N V(e^{i\theta_j})} e^{i\beta_1 X_N(\varphi) + i\gamma_1 Y_N(\varphi) - i\beta_2 X_N(\varphi') - i\gamma_2 Y_N(\varphi')} \\ &\quad = \det \Bigg(\int_0^{2\pi} e^{-i(k-l)\theta} e^{V(e^{i\theta})} \left| e^{i\theta} - e^{i\varphi} \right|^{i\beta_1} \left| e^{i\theta} - e^{i\varphi'} \right|^{-i\beta_2} e^{i\frac{(\gamma_1 - \gamma_2)}{2}\theta} e^{-\frac{i\gamma_1}{2}\varphi + \frac{i\gamma_2}{2}\varphi'} \\ &\qquad \times g_{\gamma_1}(\theta, \varphi) g_{-\gamma_2}(\theta, \varphi') \frac{d\theta}{2\pi} \Bigg)_{k,l=0}^{N-1}, \end{aligned} \tag{5.4}$$

where for $\theta, \varphi \in [0, 2\pi]$

$$g_\gamma(\theta, \varphi) = \begin{cases} e^{-i\pi\frac{\gamma}{2}}, & \theta < \varphi \\ e^{i\pi\frac{\gamma}{2}}, & \theta > \varphi \end{cases}.$$

Such determinants are known as Toeplitz determinants with Fisher–Hartwig singularities and have a long and interesting history – we refer the interested reader to e.g. [24] for a review of it. More importantly for us, there are quite recent results concerning very precise large N asymptotics of such determinants. We refer to [14] and [35] for the original results concerning asymptotics of precisely these types of determinants, and to [25] for results that contain the uniformity we require. The following theorem is a combination of [25, Theorem 1.1 and Remark 1.4] as well as [20, Theorem 1.11], though in our setting the statement of the theorem is slightly simpler, and we use slightly different notation. Also we'll only need results with either $\beta_1 = \beta_2 = 0$ or $\gamma_1 = \gamma_2 = 0$, and it's slightly simpler to formulate the results in these two case separately.

Theorem 5.1 (Deift, Its, and Krasovsky; Claeys and Krasovsky)**.** *Let V be a Laurent polynomial with a fixed degree K : $V(e^{i\theta}) = \sum_{|k|\leq K} V_k e^{ik\theta}$ and let $\beta_1, \beta_2 \in \mathbf{R}$. Then as $N \to \infty$,*

$$\begin{aligned}
&\mathbb{E}\, e^{\sum_{j=1}^N V(e^{i\theta_j})} e^{i\beta_1 X_N(\varphi) - i\beta_2 X_N(\varphi')} \\
&\quad = e^{NV_0 + \sum_{k=1}^K kV_k V_{-k}} N^{-\frac{\beta_1^2+\beta_2^2}{4}} |e^{i\varphi} - e^{i\varphi'}|^{-\frac{\beta_1\beta_2}{2}} e^{-i\frac{\beta_1}{2}(V(e^{i\varphi})-V_0) + i\frac{\beta_2}{2}(V(e^{i\varphi'})-V_0)} \\
&\quad \times \frac{G\big(1+i\frac{\beta_1}{2}\big)^2 G\big(1-i\frac{\beta_2}{2}\big)^2}{G(1+i\beta_1)G(1-i\beta_2)}(1+o(1)),
\end{aligned}$$

where G is the Barnes G-function and the error $o(1)$ is uniform in $|e^{i\varphi} - e^{i\varphi'}| > N^{\varepsilon-1}$ for any fixed $\varepsilon > 0$ (and if say $\beta_1 = 0$, then the error is uniform in φ') as well as uniform in $\{V_k\}_{|k|\le K}$ when restricted to some fixed compact subset of $\mathbf{C}^{2K+1}$.

Moreover, let $\gamma_1, \gamma_2 \in \mathbf{R}$ be such that either $\gamma_1\gamma_2 = 0$ or $|\gamma_1 + \gamma_2| < 2$ and $\pm\gamma_1, \pm\gamma_2 \notin 2\mathbf{Z}_+ = \{2, 4, ...\}$. Then as $N \to \infty$,

$$\begin{aligned}
&\mathbb{E}\, e^{\sum_{j=1}^N V(e^{i\theta_j})} e^{i\gamma_1 Y_N(\varphi) - i\gamma_2 Y_N(\varphi')} \\
&\quad = e^{NV_0 + \sum_{k=1}^K kV_k V_{-k}} N^{-\frac{\gamma_1^2+\gamma_2^2}{4}} |e^{i\varphi} - e^{i\varphi'}|^{-\frac{\gamma_1\gamma_2}{2}} e^{-\frac{\gamma_1}{2}\sum_{j=1}^K (V_j - V_{-j})e^{ij\varphi} + \frac{\gamma_2}{2}\sum_{j=1}^K (V_j - V_{-j})e^{ij\varphi'}} \\
&\quad \times G\big(1+\frac{\gamma_1}{2}\big)G\big(1-\frac{\gamma_1}{2}\big)G\big(1+\frac{\gamma_2}{2}\big)G\big(1-\frac{\gamma_2}{2}\big)(1+o(1)),
\end{aligned}$$

where again the error $o(1)$ is uniform in $|e^{i\varphi} - e^{i\varphi'}| > N^{\varepsilon-1}$ for any fixed $\varepsilon > 0$ (and if say $\gamma_1 = 0$, then the error is uniform in φ') as well as uniform in $\{V_k\}_{|k|\le K}$ when restricted to some fixed compact subset of $\mathbf{C}^{2K+1}$.

Remark 5.2. The uniformity in the coefficients of V is not explicitly stated in either [25] or [20], but this is something that can be verified as the error estimates for the jump matrices of the small norm Riemann–Hilbert problem are easily seen to be uniform in the coefficients (in the sense stated in Theorem 5.1). ■

We now turn to the proof of convergence to chaos.

5.3. **Proof of Theorem 1.7.** The proof we present here is very similar to those in [10, 58, 81]. The idea is to introduce a smoothing of the fields X_N and Y_N which comes simply from truncating the series expansion of the logarithm. Then one verifies that the approximation to the chaos coming from the truncation is close to that coming from the untruncated field in the L^2-sense through an application of Theorem 5.1. Theorem 5.1 also implies that the truncated version is close to the chaos constructed from purely Gaussian random variables.

Proof of Theorem 1.7. By a standard Cramér-Wold argument, we can reduce to the case when f is non-negative. Let us now introduce the following fields: for a fixed $M \in \mathbf{Z}_+$, let

$$X_{N,M}(\theta) = -\frac{1}{2}\sum_{k=1}^M \frac{1}{k}\left(e^{-ik\theta}\mathrm{Tr}U_N^k + e^{ik\theta}\mathrm{Tr}U_N^{-k}\right)$$

and

$$Y_{N,M}(\theta) = -\frac{1}{2i}\sum_{k=1}^M \frac{1}{k}\left(e^{-ik\theta}\mathrm{Tr}U_N^k - e^{ik\theta}\mathrm{Tr}U_N^{-k}\right).$$

Let us consider first the case of X_N. By e.g. [54, Theorem 4.28], it is then enough for us to prove the following two conditions:

(1)

$$\lim_{M\to\infty}\limsup_{N\to\infty}\mathbb{E}\left|\int_0^{2\pi}\left[\frac{e^{i\beta X_N(\theta)}}{\mathbb{E}\, e^{i\beta X_N(\theta)}} - \frac{e^{i\beta X_{N,M}(\theta)}}{\mathbb{E}\, e^{i\beta X_{N,M}(\theta)}}\right]f(\theta)d\theta\right|^2 = 0. \tag{5.5}$$

(2) If we first let $N\to\infty$ and then $M\to\infty$,

$$\int_0^{2\pi}\frac{e^{i\beta X_{N,M}(\theta)}}{\mathbb{E}\,e^{i\beta X_{N,M}(\theta)}}f(\theta)d\theta\xrightarrow{d}\mu_M(f)\xrightarrow{d}\mu(f),\tag{5.6}$$

where μ_M is some approximation to μ for which convergence can be proven.

Let us begin with the first claim. Expanding the square, we find

$$\begin{aligned}\mathbb{E}\left|\int_0^{2\pi}\left[\frac{e^{i\beta X_N(\theta)}}{\mathbb{E}\,e^{i\beta X_N(\theta)}}-\frac{e^{i\beta X_{N,M}(\theta)}}{\mathbb{E}\,e^{i\beta X_{N,M}(\theta)}}\right]f(\theta)d\theta\right|^2\\ =\int_{[0,2\pi]^2}\frac{\mathbb{E}\,e^{i\beta X_N(\varphi)}e^{-i\beta X_N(\varphi')}}{\mathbb{E}\,e^{i\beta X_N(\varphi)}\mathbb{E}\,e^{-i\beta X_N(\varphi')}}f(\varphi)f(\varphi')d\varphi d\varphi'\\ -2\int_{[0,2\pi]^2}\mathrm{Re}\frac{\mathbb{E}\,e^{i\beta X_N(\varphi)}e^{-i\beta X_{N,M}(\varphi')}}{\mathbb{E}\,e^{i\beta X_N(\varphi)}\mathbb{E}\,e^{-i\beta X_{N,M}(\varphi')}}f(\varphi)f(\varphi')d\varphi d\varphi'\\ +\int_{[0,2\pi]^2}\mathrm{Re}\frac{\mathbb{E}\,e^{i\beta X_{N,M}(\varphi)}e^{-i\beta X_{N,M}(\varphi')}}{\mathbb{E}\,e^{i\beta X_{N,M}(\varphi)}\mathbb{E}\,e^{-i\beta X_{N,M}(\varphi')}}f(\varphi)f(\varphi')d\varphi d\varphi'\end{aligned}\tag{5.7}$$

All of these terms now are of the form discussed in Theorem 5.1. Let us first focus on the first term. Setting $V=0$ and $\beta_2=0$ in Theorem 5.1, we find for each $\theta\in[0,2\pi]$, as $N\to\infty$,

$$\mathbb{E}\,e^{i\beta X_N(\theta)}=N^{-\frac{\beta^2}{4}}\frac{G(1+i\frac{\beta}{2})^2}{G(1+i\beta)}(1+o(1)).\tag{5.8}$$

In fact, from the translation invariance of the law (5.2), this expectation is constant in θ, so this statement is in particular uniform in θ. Moreover, this result could be seen directly as the expectation can be written as a variant of the Selberg integral. Combining (5.8) with another application of Theorem 5.1 with $V=0$, $\beta_1=\beta=\beta_2$ shows that as $N\to\infty$,

$$\frac{\mathbb{E}\,e^{i\beta X_N(\varphi)}e^{-i\beta X_N(\varphi')}}{\mathbb{E}\,e^{i\beta X_N(\varphi)}\mathbb{E}\,e^{-i\beta X_N(\varphi')}}=\left|e^{i\varphi}-e^{i\varphi'}\right|^{-\frac{\beta^2}{2}}(1+o(1)),$$

where the error $o(1)$ is uniform in $|e^{i\varphi}-e^{i\varphi'}|\geq N^{\varepsilon-1}$ for any fixed $\varepsilon>0$. Thus we find for the first term in (5.7)

$$\begin{aligned}&\int_{[0,2\pi]^2}\frac{\mathbb{E}\,e^{i\beta X_N(\varphi)}e^{-i\beta X_N(\varphi')}}{\mathbb{E}\,e^{i\beta X_N(\varphi)}\mathbb{E}\,e^{-i\beta X_N(\varphi')}}f(\varphi)f(\varphi')d\varphi d\varphi'\\ &=\int_{|e^{i\varphi}-e^{i\varphi'}|\leq N^{\varepsilon-1}}\frac{\mathbb{E}\,e^{i\beta X_N(\varphi)}e^{-i\beta X_N(\varphi')}}{\mathbb{E}\,e^{i\beta X_N(\varphi)}\mathbb{E}\,e^{-i\beta X_N(\varphi')}}f(\varphi)f(\varphi')d\varphi d\varphi'\\ &\quad+(1+o(1))\int_{|e^{i\varphi}-e^{i\varphi'}|\geq N^{\varepsilon-1}}|e^{i\varphi}-e^{i\varphi'}|^{-\frac{\beta^2}{2}}f(\varphi)f(\varphi')d\varphi d\varphi'\\ &=\mathcal{O}(N^{\frac{\beta^2}{2}+\varepsilon-1})+\mathcal{O}\big(N^{(\varepsilon-1)(1-\frac{\beta^2}{2})}\big)+(1+o(1))\int_{[0,2\pi]^2}|e^{i\varphi}-e^{i\varphi'}|^{-\frac{\beta^2}{2}}f(\varphi)f(\varphi')d\varphi d\varphi',\end{aligned}$$

where the implied constants in the errors can depend on f. The first error comes simply from bounding $|\mathbb{E}\,e^{i\beta X_N(\varphi)-i\beta X_N(\varphi')}|$ by one, bounding f by its maximum, and using (5.8). The second one comes from replacing the integration region $|e^{i\varphi}-e^{i\varphi'}|\geq N^{\varepsilon-1}$ with $[0,2\pi]^2$. As $\beta^2<2$, we conclude that

$$\lim_{N\to\infty}\int_{[0,2\pi]^2}\frac{\mathbb{E}\,e^{i\beta X_N(\varphi)}e^{-i\beta X_N(\varphi')}}{\mathbb{E}\,e^{i\beta X_N(\varphi)}\mathbb{E}\,e^{-i\beta X_N(\varphi')}}f(\varphi)f(\varphi')d\varphi d\varphi'=\int_{[0,2\pi]^2}\left|e^{i\varphi}-e^{i\varphi'}\right|^{-\frac{\beta^2}{2}}f(\varphi)f(\varphi')d\varphi d\varphi'.\tag{5.9}$$

Consider next the cross term in (5.5). Choosing $\beta_1 = \beta_2 = 0$ and $V(e^{i\theta}) = -\frac{i\beta}{2}\sum_{k=1}^M \frac{1}{k}(e^{ik(\theta-\varphi)} + e^{-ik(\theta-\varphi)})$ (namely $V_k = -\frac{i\beta}{2|k|}e^{-ik\varphi}$ for $1 \leq |k| \leq M$ and zero otherwise) in Theorem 5.1, we see that for a fixed M, as $N \to \infty$

$$\mathbb{E}\, e^{i\beta X_{N,M}(\varphi)} = e^{-\frac{\beta^2}{4}\sum_{k=1}^M \frac{1}{k}}(1+o(1)) \tag{5.10}$$

where the error is uniform in φ, but not necessarily in M. Combining this with (5.8) as well as another application of Theorem 5.1 with the choices $\beta_1 = \beta$, $\beta_2 = 0$, $V(e^{i\theta}) = \frac{i\beta}{2}\sum_{k=1}^M \frac{1}{k}(e^{ik(\theta-\varphi')} + e^{-ik(\theta-\varphi')})$ yields that for a fixed M, as $N \to \infty$

$$\frac{\mathbb{E}\, e^{i\beta X_N(\varphi) - i\beta X_{N,M}(\varphi')}}{\mathbb{E}\, e^{i\beta X_N(\varphi)}\mathbb{E}\, e^{-i\beta X_{N,M}(\varphi')}} = (1+o(1))e^{-i\frac{\beta}{2}V(e^{i\varphi})} = (1+o(1))e^{\frac{\beta^2}{2}\sum_{k=1}^M \frac{\cos k(\varphi-\varphi')}{k}}, \tag{5.11}$$

where the error is uniform in φ, φ'. Now it follows e.g. from [51, the proof of Lemma 6.5] that $\sum_{k=1}^M \frac{\cos k(\varphi-\varphi')}{k} \leq -\log|e^{i\varphi} - e^{i\varphi'}| + C$, for some constant C independent of M, φ, φ'. Thus combining (5.11) with a dominated convergence argument (recall that $\beta^2 < 2$) shows that

$$\begin{aligned} &\lim_{M\to\infty}\lim_{N\to\infty}\int_{[0,2\pi]^2} \frac{\mathbb{E}\, e^{i\beta X_N(\varphi) - i\beta X_{N,M}(\varphi')}}{\mathbb{E}\, e^{i\beta X_N(\varphi)}\mathbb{E}\, e^{-i\beta X_{N,M}(\varphi')}} f(\varphi)f(\varphi')d\varphi d\varphi' \\ &= \int_{[0,2\pi]^2} \lim_{M\to\infty} e^{\frac{\beta^2}{2}\sum_{k=1}^M \frac{\cos k(\varphi-\varphi')}{k}} f(\varphi)f(\varphi')d\varphi d\varphi' \\ &= \int_{[0,2\pi]^2} \left|e^{i\varphi} - e^{i\varphi'}\right|^{-\frac{\beta^2}{2}} f(\varphi)f(\varphi')d\varphi d\varphi'. \end{aligned} \tag{5.12}$$

Finally a similar argument, taking $\beta_1 = \beta_2 = 0$ and $V(e^{i\theta}) = -\frac{i\beta}{2}\sum_{k=1}^M \frac{1}{k}(e^{ik(\theta-\varphi)} + e^{-ik(\theta-\varphi)}) + \frac{i\beta}{2}\sum_{k=1}^M \frac{1}{k}(e^{ik(\theta-\varphi')} + e^{-ik(\theta-\varphi')})$ in Theorem 5.1 shows that

$$\begin{aligned} &\lim_{M\to\infty}\lim_{N\to\infty}\int_{[0,2\pi]^2} \frac{\mathbb{E}\, e^{i\beta X_{N,M}(\varphi) - i\beta X_{N,M}(\varphi')}}{\mathbb{E}\, e^{i\beta X_{N,M}(\varphi)}\mathbb{E}\, e^{-i\beta X_{N,M}(\varphi')}} f(\varphi)f(\varphi')d\varphi d\varphi' \\ &= \int_{[0,2\pi]^2} \left|e^{i\varphi} - e^{i\varphi'}\right|^{-\frac{\beta^2}{2}} f(\varphi)f(\varphi')d\varphi d\varphi'. \end{aligned} \tag{5.13}$$

Thus combining (5.9), (5.12), and (5.13) yields (5.5).

To prove (5.6), we first point out that it is a consequence of Theorem 5.1 (taking $\beta_1 = \beta_2 = 0$) that as $N \to \infty$, for a fixed M,

$$\left(\frac{\mathrm{Tr} U_N^k}{\sqrt{k}}\right)_{k=1}^M \xrightarrow{d} (Z_1, ..., Z_M)$$

where Z_i are i.i.d standard complex Gaussians (real and imaginary part independent $N(0,1/2)$ random variables). This is a remark going back to [26], where it was proven through connections to representation theory, but it can in fact be seen to be a consequence of the strong Szegő theorem. Then an application of the continuous mapping theorem (see e.g. [54, Theorem 4.27]) shows that as $N \to \infty$ (again for a fixed M and f)

$$\int_0^{2\pi} \frac{e^{i\beta X_{N,M}(\theta)}}{\mathbb{E}\, e^{i\beta X_{N,M}(\theta)}} f(\theta)d\theta \xrightarrow{d} \int_0^{2\pi} \frac{e^{\frac{i\beta}{2}\sum_{k=1}^M \frac{1}{\sqrt{k}}(e^{-ik\theta}Z_k + e^{ik\theta}Z_k^*)}}{\mathbb{E}\, e^{\frac{i\beta}{2}\sum_{k=1}^M \frac{1}{\sqrt{k}}(e^{-ik\theta}Z_k + e^{ik\theta}Z_k^*)}} f(\theta)d\theta.$$

The convergence of this to $e^{i\frac{\beta}{\sqrt{2}}X}$ (for $\beta^2 < 2$) as $M \to \infty$ then follows from recalling that the fields $\frac{1}{\sqrt{2}}\sum_{k=1}^M (e^{-ik\theta}Z_k + e^{ik\theta}Z_k^*)/\sqrt{k}$ form a standard approximation – see Example 2.9. The convergence could also be deduced from Lemma 3.5 since the sum is a truncation of the Karhunen–Loève decomposition of the underlying field. This concludes the proof for X_N. The proof for Y_N is nearly identical. We simply point out that the condition $\beta^2 < 1$ comes from the condition $|\gamma_1 + \gamma_2| < 2$ in Theorem 5.1 reducing to $|\beta| < 1$ in the case relevant to the proof. We omit further details. $\square$

Appendix A. Auxiliary results

In this appendix we record some basic facts needed to control moments of imaginary chaos near the critical point. The first one is something that gives a rough estimate required for controlling mixed moments.

Lemma A.1. *Let $U \subset \mathbf{R}^d$ be bounded and $0 < \beta < \sqrt{d}$. Then for any indices $a \geq b$ and $x_1, \dots, x_a, y_1, \dots, y_b \in U$, $a \geq b$, we have the inequality*

$$\frac{\prod_{1 \le j < k \le a} |x_j - x_k|^{\beta^2} \prod_{1 \le j < k \le b} |y_j - y_k|^{\beta^2}}{\prod_{1 \le j \le a} \prod_{1 \le k \le b} |x_j - y_k|^{\beta^2}} \le \sum_{\substack{f \colon \{1,\dots,b\} \to \{1,\dots,a\}, \\ \textit{injective}}} \frac{C}{\prod_{1 \le j \le b} |x_{f(j)} - y_j|^{\beta^2}}$$

for some constant C depending only U, a, and b – not β.

Proof. The result can be obtained by using a Gale–Shapley matching (see e.g. the appendix in [57] – we provide a proof here for the reader's convenience). For given $x_1, \dots, x_a$ and $y_1, \dots, y_b$ we may form a matching $f \colon \{1, \dots, b\} \to \{1, \dots, a\}$ via the following algorithm: Among the remaining pairs (x_j, y_k) choose one with minimal distance $|x_j - y_k|$, set $f(k) = j$, remove the points x_j and y_k from the set of remaining points and repeat. By permutation invariance of the original expression we may assume that the points matched by the algorithm are $(y_1, x_1), \dots, (y_b, x_b)$, and they are matched in this order. We may then write

$$\begin{aligned}
&\frac{\prod\limits_{1 \le j < k \le a} |x_j - x_k|^{\beta^2} \prod\limits_{1 \le j < k \le b} |y_j - y_k|^{\beta^2}}{\prod\limits_{\substack{1 \le j \le a \\ 1 \le k \le b}} |x_j - y_k|^{\beta^2}} \\
&= \frac{\prod\limits_{b+1 \le j < k \le a} |x_j - x_k|^{\beta^2}}{\prod\limits_{1 \le j \le b} |x_j - y_j|^{\beta^2}} \cdot \frac{\prod\limits_{\substack{1 \le j < k \le a \\ j \le b}} |x_j - x_k|^{\beta^2} \prod\limits_{1 \le j < k \le b} |y_j - y_k|^{\beta^2}}{\prod\limits_{\substack{1 \le j \le a \\ 1 \le k \le b \\ j \ne k}} |x_j - y_k|^{\beta^2}}.
\end{aligned}$$

We next write the second factor as

$$= \prod_{\ell=1}^{b} \left(\prod_{\ell < k \le b} \frac{|y_\ell - y_k|^{\beta^2}}{|x_k - y_\ell|^{\beta^2}} \prod_{\ell < j \le a} \frac{|x_\ell - x_j|^{\beta^2}}{|x_j - y_\ell|^{\beta^2}} \right)$$

and using the inequalities

$$\frac{|y_\ell - y_k|}{|x_k - y_\ell|} \le \frac{|y_\ell - x_k| + |x_k - y_k|}{|x_k - y_\ell|} \le 2,$$

where we use that y_k was matched before y_ℓ, and

$$\frac{|x_\ell - x_j|}{|x_j - y_\ell|} \le \frac{|x_\ell - y_\ell| + |y_\ell - x_j|}{|x_j - y_\ell|} \le 2$$

implied in turn by the fact that x_ℓ was matched before x_j, we see that

$$\frac{\prod\limits_{1 \le j < k \le a} |x_j - x_k|^{\beta^2} \prod\limits_{1 \le j < k \le b} |y_j - y_k|^{\beta^2}}{\prod\limits_{\substack{1 \le j \le a \\ 1 \le k \le b}} |x_j - y_k|^{\beta^2}} \le 2^{\beta^2 (a-1) b} \frac{\prod\limits_{b+1 \le j < k \le a} |x_j - x_k|^{\beta^2}}{\prod\limits_{1 \le j \le b} |x_j - y_j|^{\beta^2}}$$

under the assumption that the points were matched according to f. Summing over the possible matchings and bounding β^2 by d in the prefactor yields the result. $\square$

The following lemma is used for studying the behavior of imaginary multiplicative chaos near the critical point.

Lemma A.2. *Let μ be the random generalized function from Theorem 1.1. For any test function $\varphi \in C_c^\infty(U)$ we have*

$$\left|\mathbb{E}\,\mu(\varphi)^a\overline{\mu(\varphi)}^b\right| \le C(\mathbb{E}\,|\mu(\varphi)|^2)^{\min(a,b)}$$

for all integers $a, b \ge 0$ and some constant C possibly depending on φ, g from (2.1)*, a, and b, but not on β.*

Proof. By the proof of Theorem 1.3 and a direct computation

$$\left|\mathbb{E}\,\mu(\varphi)^a\overline{\mu(\varphi)}^b\right| \lesssim C_{a,b}\int_{U^{a\times b}} \frac{\prod_{1\le j<k\le a}|x_j-x_k|^{\beta^2}\prod_{1\le j<k\le b}|y_j-y_k|^{\beta^2}}{\prod_{1\le j\le a}\prod_{1\le k\le b}|x_j-y_k|^{\beta^2}}\,dx_1\dots dx_a dy_1\dots dy_b.$$

Here $C_{a,b}$ depends on φ and g, and initially also on β, since the natural estimate one uses involves terms like $e^{\beta^2\|g\|_{L^\infty(\operatorname{supp}(\varphi)\times\operatorname{supp}(\varphi))}}$, but we can always bound this from above by replacing β^2 with d, so we get a bound independent of β. We may assume that $a \ge b$, the other case is handle in the same way. It then readily follows by applying Lemma A.1 and integrating that

$$\left|\mathbb{E}\,\mu(\varphi)^a\overline{\mu(\varphi)}^b\right| \le C(\mathbb{E}\,|\mu(\varphi)|^2)^b$$

for some constant C independent of β. $\square$

Finally we conclude with a proof of Lemma 3.10.

Proof of Lemma 3.10. For fixed $x_1,\dots,x_N \in B(0,1)$, let $F\colon \{1,\dots,N\}\to\{1,\dots,N\}$ be the nearest neighbour function mapping $i \mapsto j$, where j is the index of the closest point x_j to the point x_i. By removing a set of measure 0 from $B(0,1)^N$, we may assume that F is uniquely defined. The integral then becomes

$$\sum_F \int_{U_F} e^{\frac{\beta^2}{2}\sum_{j=1}^N \log\frac{1}{\frac{1}{2}|x_j-x_{F(j)}|}}\,dx_1\dots dx_N,$$

where $U_F \subset B(0,1)^N$ is the set of those point configurations $(x_1,\dots,x_N)\in B(0,1)^N$ whose nearest neighbour function equals F. Each nearest neighbour function F can be uniquely represented by a directed graph with vertices $\{1,\dots,N\}$ and an arrow from i to $F(i)$. This graph is of the following form: It consists of $k \le \lfloor N/2\rfloor$ components, and each component consists of a 2-cycle (the two mutually closest points in the component, by the triangle inequality there can be no longer cycles) with two trees connected to the two vertices in the cycle. Without loss of generality we may assume that $(x_1,x_2),\dots,(x_{2k-1},x_{2k})$ are the vertices forming the cycles. Perform now the change of variables $u_j = \frac{1}{2}(x_j - x_{F(j)})$ for $j = 2k+1,\dots,N$, $u_1 = \frac{1}{2}(x_1-x_2)$, $u_2 = \frac{1}{2}x_2$, $\dots$, $u_{2k-1} = \frac{1}{2}(x_{2k-1}-x_{2k})$ and $u_{2k} = \frac{1}{2}x_{2k}$. Then we get the integral

$$\int_{\tilde{U}_F} \frac{2^N}{|u_1|^{\beta^2}|u_3|^{\beta^2}\dots|u_{2k-1}|^{\beta^2}|u_{2k+1}|^{\beta^2/2}\dots|u_N|^{\beta^2/2}}\,du_1\dots du_N$$

for some new integration domain $\tilde{U}_F$. We have $|u_j|\le 1$ for all j and moreover the balls $B_j = \{y \in \mathbf{R}^d : |y-x_j| \le |u_j|\}$, $j = 1,3,\dots,2k-1,2k+1,2k+2,\dots,N$ are disjoint (since $|u_j|$ is half the distance from x_j to its nearest neighbour). Each such ball is contained in $B(0,2)$, and thus by comparing volumes we get the inequality

$$|u_1|^d + |u_3|^d + \dots + |u_{2k-1}|^d + |u_{2k+1}|^d + \dots + |u_N|^d \le 2^d.$$

In particular the new integration domain $\tilde{U}_F$ is contained in

$$\{|u_1|^d + |u_3|^d + \dots + |u_{2k-1}|^d + |u_{2k+1}|^d + \dots + |u_N|^d \le 2^d, |u_2|,\dots,|u_{2k}|\le 1\}.$$

Hence we get the upper bound

$$\begin{aligned}
&\int_{\tilde{U}_F} \frac{2^N}{|u_1|^{\beta^2}|u_3|^{\beta^2}\dots|u_{2k-1}|^{\beta^2}|u_{2k+1}|^{\beta^2/2}\dots|u_N|^{\beta^2/2}}\,du_1\dots du_N\\
&\le c^N \int_{(\partial B(0,1))^{N-k}}\int_{r_1^d+\dots+r_{N-k}^d\le 2^d} r_1^{-\beta^2+d-1}\dots r_k^{-\beta^2+d-1} r_{k+1}^{-\frac{\beta^2}{2}+d-1}\dots r_{N-k}^{-\frac{\beta^2}{2}+d-1}\,dr_1\dots dr_{N-k}\\
&\le c^N \int_{t_1+\dots+t_{N-k}\le 1} t_1^{-\frac{\beta^2}{d}}\dots t_k^{-\frac{\beta^2}{d}} t_{k+1}^{-\frac{\beta^2}{2d}}\dots t_{N-k}^{-\frac{\beta^2}{2d}}\,dt_1\dots dt_{N-k}\\
&\le c^N \frac{\Gamma(1-\frac{\beta^2}{d})^k\Gamma(1-\frac{\beta^2}{2d})^{N-2k}}{\Gamma(k(1-\frac{\beta^2}{d})+(N-2k)(1-\frac{\beta^2}{2d}))}\int_0^1 t^{N-k-k\frac{\beta^2}{d}-(N-2k)\frac{\beta^2}{2d}-1}\,dt\\
&\le \frac{c^N}{\Gamma(k(1-\frac{\beta^2}{d})+(N-2k)(1-\frac{\beta^2}{2d})+1)},
\end{aligned}$$

where c is some constant that may get bigger on each line of the above and following computations, and which is allowed to depend on β^2 and d but not on N or k. Above we used Dirichlet's integral formula, see e.g. [82, Section 12.5]. Thus we have

$$\int_{U_F} e^{\frac{\beta^2}{2}\sum_{j=1}^N \log\frac{1}{\frac{1}{2}|x_j-x_{F(j)}|}}\,dx_1\dots dx_N \le \frac{c^N}{\Gamma(N(1-\frac{\beta^2}{2d})-k+1)}, \tag{A.1}$$

where the right hand side only depends on F via the number of components in the directed graph associated with F.

Next we bound the number of nearest neighbour functions whose graphs have k components. As already mentioned above, each component consists of a 2-cycle augmented with two trees, or a simpler way to think of them might be as unordered pairs of rooted trees whose roots form the cycle. It is worth noting that the map from the nearest neighbour functions to their associated graphs is not a surjection since geometrical reasons limit the number of incoming edges each vertex may have. However, since we are only concerned with an upper bound, we will ignore this fact and simply count all possible labeled graphs with N vertices and k components of the above prescribed type, with labels corresponding to the variables $x_1,\dots,x_N$. This is a fairly straightforward task to which standard counting methods using generating functions apply. Here we have written the argument using combinatorial species, see for example [11] for an introduction to the subject. For an argument formulated in more elementary terms, we refer to [45]. Let E_k be the species of (unordered) sets of k elements and let T be the species of rooted trees. The species of a single component in the graph is then $E_2\circ T$ (an unordered pair of rooted trees, whose roots correspond to the cycle). A set of k of these gives us then the required species G_k of nearest neighbour graphs with k components, $G_k = E_k\circ(E_2\circ T)$. The labeled generating function of E_k is given by $E_k(x)=\frac{x^k}{k!}$ and hence

$$G_k(x)=\frac{(T(x)^2/2)^k}{k!}=\frac{T(x)^{2k}}{2^k k!}.$$

The species T itself satisfies the equation $T = X\cdot(E\circ T)$, where E is the species of sets (a rooted tree consists of a root and a set of subtrees). Since $E(x)=e^x$, the labeled generating function of T satisfies the equation $T(x)=xe^{T(x)}$. In particular, if we let $f(x)=xe^{-x}$, then f is the compositional inverse of T, and we may use the Lagrange inversion formula to compute for $N\ge 2k$ that

$$[x^N]T(x)^{2k}=\frac{2k}{N}[x^{-2k}]f(x)^{-N}=\frac{2k}{N}[x^{-2k}]\frac{e^{Nx}}{x^N}=\frac{2k}{N}[x^{-2k}]\sum_{j=0}^\infty\frac{N^jx^{j-N}}{j!}=\frac{2kN^{N-2k-1}}{(N-2k)!},$$

where $[x^k]g(x)$ is the coefficient of x^k in some power series g. Hence the number of nearest neighbour graphs with N vertices and k components (ignoring the geometrical restrictions) is

$$\frac{N!2kN^{N-2k-1}}{2^kk!(N-2k)!}\le c^N\frac{N!}{k!}\le c^N(N-k)!, \tag{A.2}$$

where the first inequality follows by Stirling's approximation and the second follows from the fact that $\binom{N}{k} \leq 2^N$.

The proof is easily finished by combining (A.1) and (A.2) with another application of Stirling:

$$\begin{aligned}
&\int_{B(0,1)^N} \exp\Big(\frac{\beta^2}{2}\sum_{j=1}^N \log\frac{1}{\frac{1}{2}\min_{k\neq j}|x_j-x_k|}\Big)\,dx_1\ldots dx_N \\
&= \sum_F \int_{U_F} e^{\frac{\beta^2}{2}\sum_{j=1}^N \log\frac{1}{\frac{1}{2}|x_j-x_{F(j)}|}}\,dx_1\ldots dx_N \\
&\leq c^N \sum_{k=1}^{\lfloor N/2\rfloor} \frac{(N-k)!}{\Gamma(N(1-\frac{\beta^2}{2d})-k+1)} \leq c^N \sum_{k=1}^{\lfloor N/2\rfloor} N^{N\frac{\beta^2}{2d}} \leq c^N N^{N\frac{\beta^2}{2d}},
\end{aligned}$$

where again the value of c may not be same in each of the places it appears. □

References

[1] L.-P. Arguin, D. Belius, and P. Bourgade: Maximum of the Characteristic Polynomial of Random Unitary Matrices. Comm. Math. Phys. 349 (2017), no. 2, 703–751.

[2] L.-P. Arguin, D. Belius, P. Bourgade, M. Radziwiłł, and K. Soundararajan: Maximum of the Riemann zeta function on a short interval of the critical line. To appear in Commun. Pure Appl. Math. Preprint arXiv:1612.08575.

[3] K. Astala, P. Jones, A. Kupiainen, and E. Saksman: Random conformal weldings. Acta Math. 207 (2011), no. 2, 203–254.

[4] N. Aronszajn and K.T. Smith: Theory of Bessel potentials. I. Ann. Inst. Fourier (Grenoble) 11 (1961) 385–475.

[5] R. Adler and J. Taylor: Random fields and geometry. Springer 2007.

[6] J. Barral, X. Jin, and B. Mandelbrot: Uniform convergence for complex $[0,1]$-martingales. Ann. Appl. Probab., 2010, vol. 20, no 4, 1205–1218.

[7] J. Barral and B. Mandelbrot: Fractional multiplicative processes. Ann. Inst. H. Poincaré. 45 (2009) no. 4, 1116–1129.

[8] R. Baxter: Exactly solved models in statistical mechanics. Academic Press, London/New York, 1982.

[9] N. Berestycki: An elementary approach to Gaussian multiplicative chaos. Electron. Commun. Probab. 22 (2017), Paper No. 27, 12 pp.

[10] N. Berestycki, C. Webb, and M.D. Wong: Random Hermitian Matrices and Gaussian Multiplicative Chaos. Probab. Theory Relat. Fields (2017). https://doi.org/10.1007/s00440-017-0806-9

[11] F. Bergeron, G. Labelle, and P. Leroux: Combinatorial Species and Tree-like Structures. Cambridge University Press (1998).

[12] A. Borodin and P. Ferrari: Anisotropic growth of random surfaces in 2+1 dimensions. Comm. Math. Phys. 325 (2014), no. 2, 603–684.

[13] C. Boutillier and B. de Tilière: Height representation of XOR-Ising loops via bipartite dimers. Electron. J. Probab. 19 (2014) no. 80. 1–33.

[14] A. Böttcher and B. Silbermann: Toeplitz matrices and determinants with Fisher-Hartwig symbols. J. Funct. Anal. 63 (1985), 178–214

[15] F. Camia, C. Garban, and C. Newman: Planar Ising magnetization field I. Uniqueness of the critical scaling limit. Ann. Probab. 43 (2015) no. 2 528–571.

[16] F. Camia, C. Garban, and C. Newman: Planar Ising magnetization field II. Properties of the critical and near-critical scaling limits. Ann. Inst. Henri Poincaré (B) 52 (2016), no. 1, 146–161.

[17] D. Carpentier and P. Le Doussal: Glass transition of a particle in a random potential, front selection in nonlinear RG and entropic phenomena in Liouville and Sinh-Gordon models. Phys. Rev. E 63:026110 (2001).

[18] D. Chelkak, C. Hongler, and K. Izyurov: Conformal invariance of spin correlations in the planar Ising model, Ann. of Math. (2) 181 (2015), no. 3, 1087–1138.

[19] R. Chhaibi, T. Madaule, and J. Najnudel: On the maximum of the CβE field. Duke Math. J. 167 (2018), no. 12, 2243–2345.
[20] T. Claeys and I. Krasovsky: Toeplitz determinants with merging singularities. Duke Math. J. 164 (2015), no. 15, 2897–2987.
[21] S. Coleman: Quantum sine-Gordon equation as the massive Thirring model. Phys. Rev. D 11 (1975), 2088.
[22] F. David, A. Kupiainen, R. Rhodes, and V. Vargas: Liouville Quantum Gravity on the Riemann sphere. Commun. Math. Phys. 342 (3), 869–907 (2016)
[23] P. Deift, A. Its, and I. Krasovsky: Asymptotics of Toeplitz, Hankel, and Toeplitz+Hankel determinants with Fisher–Hartwig singularities. Ann. of Math. (2) 174 (2011), no. 2, 1243–1299
[24] P. Deift, A. Its, and I. Krasovsky: Toeplitz matrices and Toeplitz determinants under the impetus of the Ising model: some history and some recent results. Comm. Pure Appl. Math. 66 (2013), no. 9, 1360–1438.
[25] P. Deift, A. Its, and I. Krasovsky: On the asymptotics of a Toeplitz determinant with singularities. Random matrix theory, interacting particle systems, and integrable systems, 93–146, Math. Sci. Res. Inst. Publ., 65, Cambridge Univ. Press, New York, 2014.
[26] P. Diaconis and M. Shashahani: On the eigenvalues of random matrices. Studies in applied probability. J. Appl. Probab. 31A (1994), 49–62.
[27] P. Di Francesco, P. Mathieu, and D. Sénéchal: Conformal field theory. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997.
[28] J. Dimock and T. Hurd: Sine-Gordon Revisited. Ann. Henri Poincaré 1, no. 3, (2000), 499–541.
[29] J. Ding, R. Roy, and O. Zeitouni: Convergence of the centered maximum of log-correlated Gaussian fields. Ann. Probab. 45 (2017), no. 6A, 3886–3928.
[30] J. Dubédat: SLE and the free field: partition functions and couplings. J. Amer. Math. Soc. 22 (2009), no. 4, 995–1054.
[31] J. Dubédat: Exact bosonization of the Ising model. Preprint arXiv:1112.4399.
[32] B. Duplantier and S. Sheffield: Liouville quantum gravity and KPZ. Invent. Math. 185 (2011), no. 2, 333–393.
[33] R. Durrett: Probability: Theory and Examples. Fourth edition. Cambridge University Press 2010.
[34] F. Dyson and A. Lenard: Stability of Matter. I. J. Math. Phys. 8 (1967), 423–434.
[35] T. Ehrhardt: A status report on the asymptotic behavior of Toeplitz determinants with Fisher-Hartwig singularities. Operator Theory: Adv. Appl. 124, 217–241 (2001).
[36] C. Fefferman and R. de la Lleve: Relativistic Stability of Matter - I. Revista Matemática Iberoamericana 2 (1986), no. 2, 119–213.
[37] J. Frölich: Classical and Quantum Statistical Mechanics in One and Two Dimensions: Two-Component Yukawa and Coulomb Systems, Commun. Math. Phys. 47, 233–268 (1976).
[38] M. Furlan and J.-C. Mourrat: A tightness criterion for random fields, with application to the Ising model. Electron. J. Probab. 22 (2017), Paper no. 97, 29 pp.
[39] Y.V. Fyodorov, G.A. Hiary, and J.P. Keating: Freezing Transition, Characteristic Polynomials of Random Matrices, and the Riemann Zeta-Function. Phys. Rev. Lett. 108, 170601 (2012)
[40] Y.V. Fyodorov and J.P. Keating: Freezing transitions and extreme values: random matrix theory, and disordered landscapes. Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 372 (2014), no. 2007, 20120503, 32 pp.
[41] Y.V. Fyodorov, B.A. Khoruzhenko, and N.J. Simm: Fractional Brownian motion with Hurst index $H = 0$ and the Gaussian unitary ensemble. Ann. Probab. 44 (2016), no. 4, 2980–3031.
[42] J. Garnett and D. Marshall: Harmonic measure. Cambridge University Press 2008.
[43] J. Glimm and A. Jaffe: Quantum physics: a functional integral point of view. Springer-Verlag, Berlin and New York, 1981.
[44] L. Grafakos: Modern Fourier analysis. Third edition. Graduate Texts in Mathematics, 250. Springer, New York, 2014.
[45] J. Gunson, and L. S. Panta: Two-dimensional neutral Coulomb gas. Comm. Math. Phys. 52 (1977), no. 3, 295–304.

[46] M. Hairer, H. Shen: The Dynamical Sine-Gordon Model. Comm. Math. Phys. 341 (2016), no. 3, 933–989.
[47] C.P. Hughes, J.P. Keating, and N. O'Connell: On the characteristic polynomial of a random unitary matrix. Comm. Math. Phys. 220 (2001), no. 2, 429–451.
[48] T. Hytönen, J. Van Neerven, M. Veraar, L. Weis: Analysis in Banach spaces. Vol. I. Martingales and Littlewood–Paley theory. Springer, Cham, 2016.
[49] C. Itzykson and J. Zuber: Quantum field theory and the two-dimensional Ising model. Phys. Rev. D, 15:2875–2884 (1977).
[50] V. Ivanov and G. Olshanski: Kerovs central limit theorem for the Plancherel measure on Young diagrams. Symmetric functions 2001: surveys of developments and perspectives, pp. 93–151. Kluwer Acad. Publ, Dordrecht (2002)
[51] J. Junnila and E. Saksman: Uniqueness of critical Gaussian chaos. Electron. J. Probab. 22 (2017), no. 11, 31 pp.
[52] J. Junnila, E. Saksman and C. Webb: Decompositions of log-correlated fields with applications. Preprint arXiv:1808.06838.
[53] J.-P. Kahane: Sur le chaos multiplicatif. Ann. Sci. Math. Québec 9(2), 105–150 (1985)
[54] O. Kallenberg: Foundations of modern probability. Second edition. Probability and its Applications (New York). Springer-Verlag, New York, 2002. xx+638 pp.
[55] R. Kenyon: Dominos and the Gaussian free field. Ann. Probab. 29(3), 1128–1137 (2001)
[56] A. Kupiainen, R. Rhodes, and V. Vargas: Integrability of Liouville theory: proof of the DOZZ Formula. Preprint arXiv:1707.08785.
[57] H. Lacoin, R. Rhodes, and V. Vargas: Complex gaussian multiplicative chaos. Comm. Math. Phys. 337 (2015), 569–632.
[58] G. Lambert, D. Ostrovsky, and N. Simm: Subcritical Multiplicative Chaos for Regularized Counting Statistics from Random Matrix Theory. Commun. Math. Phys. (2018). https://doi.org/10.1007/s00220-018-3130-z
[59] G. Lambert and E. Paquette: Probab. Theory Relat. Fields (2018). https://doi.org/10.1007/s00440-018-0832-2
[60] G. Lawler, X. Sun, and W. Wu: Loop-erased random walk, uniform spanning forests and bi-Laplacian Gaussian field in the critical dimension. Preprint arXiv:1608.02987.
[61] T. Leblé, S. Serfaty, O. Zeitouni: Large Deviations for the Two-Dimensional Two-Component Plasma. Comm. Math. Phys. 350 (2017), no. 1, 301–360.
[62] S. Lukyanov and A. Zamolodchikov: Exact expectation values of local fields in quantum sine-Gordon model. Nucl.Phys. B493 (1997) 571–587.
[63] T. Madaule: Maximum of a log-correlated Gaussian field. Ann. Inst. Henri Poincaré Probab. Stat. 51 (2015), no. 4, 1369–1431.
[64] B. M. McCoy and T. T. Wu: The Two-Dimensional Ising Model. Harvard University Press 1973.
[65] Y. Meyer: Wavelets and operators. I Wavelets. Hermann, Paris, 1990.
[66] J. Miller and S. Sheffield: Imaginary geometry I: interacting SLEs. Probab. Theory Related Fields 164 (2016), no. 3–4, 553–705.
[67] J. Miller and S. Sheffield: Liouville quantum gravity and the Brownian map I: The QLE(8/3,0) metric. Preprint arXiv:1507.00719.
[68] J. Najnudel: On the extreme values of the Riemann zeta function on random intervals of the critical line. Probab. Theory Relat. Fields (2017). https://doi.org/10.1007/s00440-017-0812-y
[69] C. Newman and W. Wu: Lee–Yang Property and Gaussian Free Fields. Preprint arXiv:1708.08820.
[70] F. Nicolò, J. Renn, A. Steinmann: On the massive sine-Gordon equation in all regions of collapse. Comm. Math. Phys. 105 no. 2 (1986), 291–326.
[71] L. Onsager: Electrostatic interaction between molecules. J. Phys. Chem. 43 (1939), 189–196.
[72] E. Paquette and O. Zeitouni: The maximum of the CUE field. Int. Math. Res. Not. 1–92 (2017)
[73] R. Rhodes and V. Vargas: Gaussian multiplicative chaos and applications: a review. Probab. Surv. 11 (2014), 315–392.

[74] B. Rider and B. Virág: The noise in the circular law and the Gaussian free field. Int. Math. Res. Not. IMRN 2007, no. 2, Art. ID rnm006, 33 pp.
[75] E. Saksman and C. Webb: The Riemann zeta function and Gaussian multiplicative chaos: statistics on the critical line. Preprint arXiv:1609.00027
[76] S. Sheffield: Gaussian free fields for mathematicians. Probab. Theory Related Fields 139 (2007), no. 3–4, 521–541.
[77] S. Sheffield: Conformal weldings of random surfaces: SLE and the quantum gravity zipper. Ann. Probab. 44(5), 3474–3545 (2016)
[78] B. Simon (1979): Functional integration and quantum physics. Pure and Applied Mathematics, 86. Academic Press, New York-London, 1979
[79] J. P. Solovej: Some simple counterexamples to the Onsager lemma on metric spaces. Rep. Math. Phys. 28 (1989), no. 3, 335–338.
[80] H. Triebel: Theory of function spaces. Mathematik und ihre Anwendungen in Physik und Technik, 38. Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1983.
[81] C. Webb: The characteristic polynomial of a random unitary matrix and Gaussian multiplicative chaos – the L^2-phase. Electron. J. Probab. 20 (2015), no. 104, 21 pp.
[82] E. T. Whittaker and G. N. Watson: A Course of Modern Analysis. Cambridge University Press; 4th edition (1927).
[83] D. B. Wilson: XOR-Ising loops and the Gaussian free field. Preprint arXiv:1102.3782.
[84] A. Zamolodchikov: Integrable field theory from conformal field theory. Adv. Stud. Pure Math. 19 (1989) 641–674.
[85] A. Zamolodchikov and Al. Zamolodchikov: Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models. Ann. Phys. 120, 253–291 (1979)

University of Helsinki, Department of Mathematics and Statistics, P.O. Box 68, FIN-00014 University of Helsinki, Finland
Email address: janne.junnila@helsinki.fi

Department of Mathematical Sciences, Norwegian University of Science and Technology (NTNU), NO-7491 Trondheim, Norway

University of Helsinki, Department of Mathematics and Statistics, P.O. Box 68, FIN-00014 University of Helsinki, Finland
Email address: eero.saksman@helsinki.fi

Department of mathematics and systems analysis, Aalto University, P.O. Box 11000, 00076 Aalto, Finland
Email address: christian.webb@aalto.fi