Introduction to Complex Analysis

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Contents

Notation

Set theory

$x \in A$	x is an element of the set A
$A \cup B$	union of two sets
$A \cap B$	intersection of two sets
$A \setminus B$	difference of two sets
$A \Delta B$	symmetric difference: $A \Delta B = (A \setminus B) \cup (B \setminus A)$
$\mathcal{P}(A)$	power set of A (the set of all subsets of A)
$(x_n)_{n=1}^{N}$	finite sequence $x_1, x_2, \ldots, x_N$
$(x_n)_{n=1}^{\infty}$	infinite sequence $x_1, x_2, \ldots$
$(x_n)_n$	countable (finite or infinite) family
$(x_i)_{i \in I}$	family indexed by an arbitrary index set I
$\bigcup_{i \in I} A_i$	union of a family of sets
$\bigcap_{i \in I} A_i$	intersection of a family of sets
$\biguplus_{i \in I} A_i$	union of disjoint sets
$\lvert A \rvert$	number of elements in the set A

Sets of numbers

$\mathbb{N}$	natural numbers $1, 2, \ldots$
$\mathbb{Z}$	integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\hat{\mathbb{C}}$	Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$
$i\mathbb{R}$	imaginary numbers
$\mathbb{D}$	open unit disc $\mathbb{D} = \{z \in \mathbb{C} : \lvert z \rvert < 1\}$
$B(z, r)$	open ball $B(z, r) = \{w \in \mathbb{C} : \lvert w - z \rvert < r\}$
$A(z, r, R)$	annulus $A(z, r, R) = \{w \in \mathbb{C} : r < \lvert w - z \rvert < R\}$

Complex analysis

$\mathrm{Re}(z)$	real part, $\mathrm{Re}(x+iy)=x$
$\mathrm{Im}(z)$	imaginary part, $\mathrm{Im}(x+iy)=y$
$\lvert z\rvert$	modulus, $\lvert z\rvert=\sqrt{x^2+y^2}$
$\overline{z}$	conjugate $\overline{z}=x-iy$
$\mathrm{Arg}(z)$	principal branch of argument, $\mathrm{Arg}(z)\in(-\pi,\pi]$
$\mathrm{Log}(z)$	principal branch of logarithm, $\mathrm{Im}(\mathrm{Log}(z))\in(-\pi,\pi]$
$\alpha\wedge\beta$	exterior product of α and β
$d\omega$	exterior derivative of ω

Usual conventions

U	open set $U\subset\mathbb{C}$
z, w	complex numbers
x, y	real coordinates, $z=x+iy$
γ	curve $\gamma\colon[0,1]\to\mathbb{C}$

Introduction

The aim of these lecture notes is to present the basics of complex analysis with the main points of focus being contour integrals and power series. More broadly, complex numbers provide a flexible framework to do vector analysis in the plane, and to fully leverage their power in computations we have opted to present our complex calculus in a modern way using differential forms. In this way, many of the classical complex analysis results such as Cauchy's integral theorem are seen to be simple consequences of Stokes' theorem and hold not only for holomorphic functions but for arbitrary closed 1-forms.

The lecture notes have been written from scratch without following any existing text books or lecture notes in particular, but some sources of inspiration have been the earlier lecture notes by Hurri–Syrjänen [2] and the text books by Rudin and Spivak [4, 5] for differential forms.

Complex numbers

1.1 *Complex arithmetic*

The history of complex numbers is closely linked with the history of solving polynomial equations. The third order equation $x^3 + px + q = 0$ for given $p, q \in \mathbb{R}$ can be solved using Cardano's formula

$$x = A - \frac{p}{3A}, \quad A = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

When $\frac{q^2}{4} + \frac{p^3}{27} \geq 0$, this can be computed using real numbers to get a real solution of the equation. On the other hand, when $\frac{q^2}{4} + \frac{p^3}{27} < 0$ it turns out that the roots are all real, yet cannot be expressed using only real-valued square and cubic roots. This eventually led to the expansion of the number system by introducing imaginary numbers such as $i = \sqrt{-1}$. A modern algebraic definition of complex numbers from this point of view would be the following.

Definition 1.1. The ring $\mathbb{C} := \mathbb{R}[i]/(i^2 + 1)$ is called the ring of **complex numbers**. ◆

Here $\mathbb{R}[i]$ denotes the ring of real polynomials of the symbolic variable i, and $(i^2 + 1)$ denotes the ideal generated by the polynomial $i^2 + 1$. This means that in the quotient ring $\mathbb{R}[i]/(i^2 + 1)$ the equation $i^2 + 1 = 0$ holds and hence i is a square root of -1.

Example 1.2. If $x_1 + y_1 i \in \mathbb{C}$ and $x_2 + y_2 i \in \mathbb{C}$, then their product is

$$(x_1+y_1 i)(x_2+y_2 i) = x_1x_2+x_1y_2i+y_1x_2i+y_1y_2i^2 = x_1x_2-y_1y_2+(x_1y_2+y_1x_2)i.$$

In particular $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1.$ ◆

By using the relation $i^2 = -1$ any polynomial $a_n i^n + a_{n-1} i^{n-1} + \cdots + a_1 i + a_0$ can be reduced to a polynomial of the form $x + iy$ for some $x, y \in \mathbb{R}$. Moreover, $x_1 + iy_1 = x_2 + iy_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$. This means that we can also define $\mathbb{C}$ as ordered pairs of real numbers, or equivalently as points in the Euclidean plane $\mathbb{R}^2$.

Definition 1.3 (Alternative definition of $\mathbb{C}$)**.** The ring $\mathbb{C}$, also known as the **complex plane**, is defined by endowing the vector space $\mathbb{R}^2$ with the multiplication operation $(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)$. We write $(x, y) = x + iy$, treating $1 = (1, 0)$ and $i = (0, 1)$ as basis vectors. ◆

Lemma 1.4. *The two definitions give rise to isomorphic rings via the isomorphism $x + iy \mapsto (x, y)$.*

Proof. Exercise. □

Remark. Although it does not matter whether one uses Definition 1.1 or 1.3, for the purposes of this course the second one is perhaps more natural since, as we will see, complex analysis is tightly connected with the geometry and topology of the Euclidean plane. The advantages of the first definition are that it is historically motivated, and that if one knows about polynomial rings and their quotients, it immediately tells us that $\mathbb{C}$ is a commutative ring. ◆

Definition 1.5. If $z = x + iy$ is a complex number (with $x, y \in \mathbb{R}$), then x is called the **real part** of z and denoted by $\mathrm{Re}(z)$, while y is called the **imaginary part** of z and denoted by $\mathrm{Im}(z)$. ◆

In particular $\mathbb{R}$ is embedded in $\mathbb{C}$ via the mapping $x \mapsto x + 0i$. The sets $\mathbb{R} = \{z \in \mathbb{C} : \mathrm{Im}(z) = 0\}$ and $i\mathbb{R} = \{z \in \mathbb{C} : \mathrm{Re}(z) = 0\}$ are called the **real and imaginary axes**, respectively.

Definition 1.6. The **modulus** of a complex number $z \in \mathbb{C}$ is its Euclidean norm $|z| := \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} \in [0, \infty)$. ◆

We recall that norms satisfy the triangle inequality $|z + w| \leq |z| + |w|$.

Definition 1.7. The **complex conjugate** of a complex number $z = x + iy$ is given by $\overline{z} := x - iy$. ◆

We have illustrated the complex plane, the real and imaginary axes, a complex number z, its conjugate $\overline{z}$ and modulus $|z|$ in Figure 1.1.

A useful property of the complex conjugate is that it can be used to factor $|z|^2$, since for $z = x + iy$ we have

$$z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2.$$

We can use this to show that $\mathbb{C}$ is in fact a **field**, meaning that every $z \in \mathbb{C} \setminus \{0\}$ has a multiplicative inverse. Indeed, note that since $\mathbb{R}$ is a field, the inverse of $|z|^2$ exists and hence $\overline{z}/|z|^2$ is an inverse of z.

Example 1.8. The inverse of $1 + 2i$ is

$$\frac{1}{1 + 2i} = \frac{1 - 2i}{|1 + 2i|^2} = \frac{1 - 2i}{1^2 + 2^2} = \frac{1}{5} - \frac{2}{5}i.$$ ◆

In the following lemma we have listed some basic identities regarding conjugates, real and imaginary parts and the modulus.

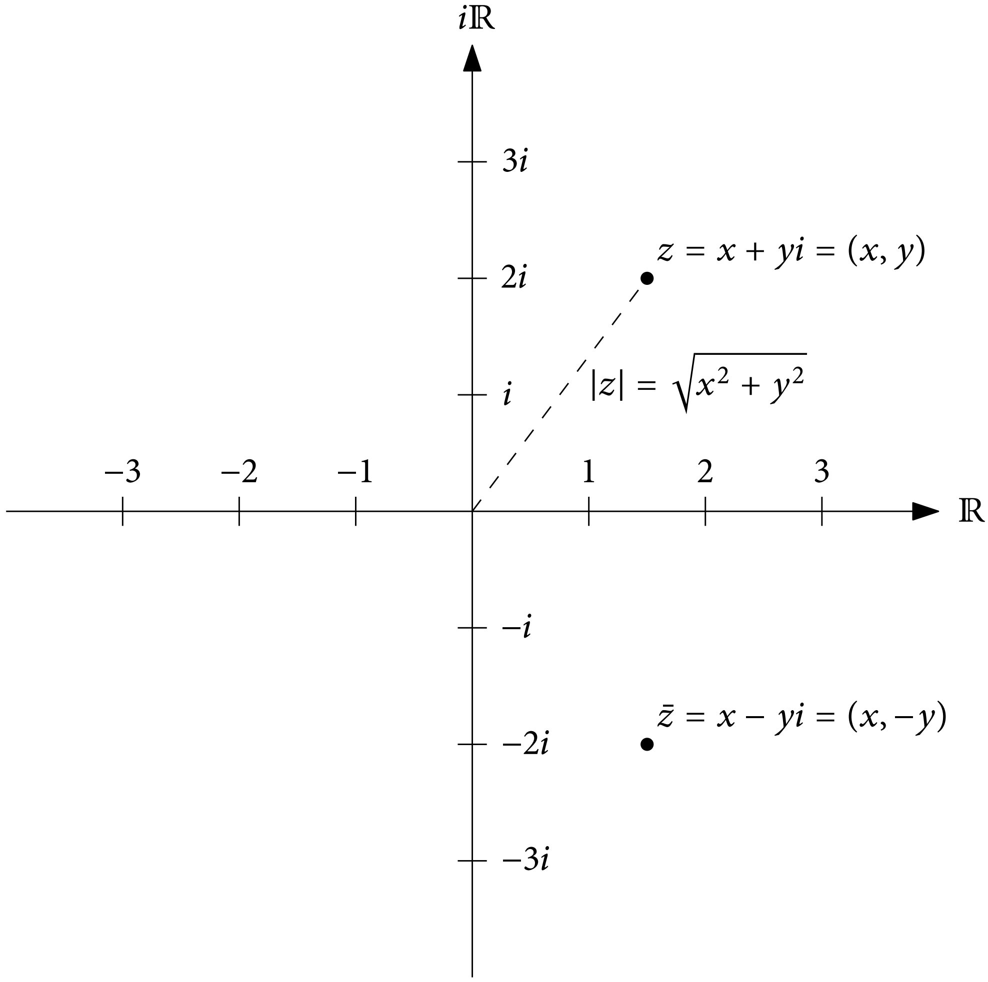

Figure 1.1: The complex plane.

Lemma 1.9. *The following hold for all $z, w \in \mathbb{C}$:*

- $\overline{\overline{z}} = z$
- $\overline{z + w} = \overline{z} + \overline{w}$
- $\overline{zw} = \overline{z} \cdot \overline{w}$
- $\overline{(z/w)} = \overline{z}/\overline{w}$ *(assuming $w \neq 0$)*
- $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$
- $\mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z})$
- $|\overline{z}| = |z|$
- $|zw| = |z||w|$

Proof. Exercise. □

1.2 *Convergent sequences and series*

In Definition 1.3 we interpreted complex numbers as points in the plane. This also lets us use the Euclidean topology of $\mathbb{R}^2$ as our topology in $\mathbb{C}$. In particular, we use the modulus $|\cdot|$ to define the **distance** $|z - w|$ between two points $z, w \in \mathbb{C}$, which makes $\mathbb{C}$ a complete metric space. We denote by

$$B(z, r) := \{w \in \mathbb{C} : |w - z| < r\}$$

the open ball of radius $r > 0$ around $z \in \mathbb{C}$.

Theorem 1.10. *The operations $(z, w) \mapsto z + w$, $(z, w) \mapsto zw$, $z \mapsto \overline{z}$, $z \mapsto |z|$ and $\mathbb{C} \setminus \{0\} \ni z \mapsto z^{-1}$ are continuous.*

Proof. Let us check $(z, w) \mapsto zw$ and leave the rest as an exercise. Note that $(z, w) \mapsto zw$ can be viewed as a map from $(\mathbb{R}^2)^2 \cong \mathbb{R}^4 \to \mathbb{R}^2$ given by $(x_1, y_1, x_2, y_2) \mapsto (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$, which is continuous since sums and products of real numbers are continuous. (Notice that we used here the fact that the product topology of $\mathbb{R}^2 \times \mathbb{R}^2$ is the same as the topology of $\mathbb{R}^4$.) □

As usual, a sequence $(z_k)_{k \geq 1}$ of complex numbers converges to z if for every $\varepsilon > 0$ there exists $k_0 \geq 1$ such that $|z - z_k| < \varepsilon$ for all $k \geq k_0$. This is equivalent to asking that the coordinate sequences $(\mathrm{Re}(z_k))_{k \geq 1}$ and $(\mathrm{Im}(z_k))_{k \geq 1}$ converge to $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$.

Definition 1.11. If $(z_k)_{k \geq 1}$ are complex numbers such that the sequence $S_n = \sum_{k=1}^{n} z_k$ converges to $S \in \mathbb{C}$, we say that the series $\sum_{k=1}^{\infty} z_k$ converges and equals S. ◆

Again, convergence of a series $\sum_{k=1}^{\infty} z_k$ is equivalent to the convergence of $\sum_{k=1}^{\infty} \operatorname{Re}(z_k)$ and $\sum_{k=1}^{\infty} \operatorname{Im}(z_k)$.

Definition 1.12. We say that a series $\sum_{k=1}^{\infty} z_k$ **converges absolutely** if the series $\sum_{k=1}^{\infty} |z_k|$ converges. ◆

Lemma 1.13. *If a series converges absolutely, then the series converges.*

Proof. Since $\mathbb{C}$ is complete, it is enough to show that $\sum_{k=1}^{n} z_k$ is a Cauchy sequence. For $n \geq m$ we have by the triangle inequality

$$\left|\sum_{k=1}^{n} z_k - \sum_{k=1}^{m} z_k\right| = \left|\sum_{k=m+1}^{n} z_k\right| \leq \sum_{k=m+1}^{n} |z_k|$$

which goes to 0 as $n, m \to \infty$ since by assumption $\sum_{k=1}^{n} |z_k|$ converges and is therefore Cauchy. □

One useful application of absolute convergence is to the exchange of order of summation.

Theorem 1.14. *Let $(z_{j,k})_{j,k=1}^{\infty}$ be a double sequence of complex numbers. Suppose that $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |z_{j,k}| < \infty$. Then $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} z_{j,k} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} z_{j,k}$ and both series converge.*

Proof. Writing $z_{j,k} = x_{j,k} + i y_{j,k}$ and noticing that $|x_{j,k}| \leq |z_{j,k}|$, we see that $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |x_{j,k}| < \infty$. By the corresponding theorem for real double sequences we then get

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} x_{j,k} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} x_{j,k}$$

with both series converging. Similarly, we get

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} y_{j,k} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} y_{j,k}$$

and it follows that

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} z_{j,k} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} z_{j,k}.$$ □

1.3 *The exponential function*

Definition 1.15. A series of the form $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ where $a_k, z_0, z \in \mathbb{C}$ is called a **power series** with coefficients a_k and variable z, centered at z_0. ◆

A power series $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ defines a function $f : D \to \mathbb{C}$ on its domain of convergence $D = \{z \in \mathbb{C} : \sum_{k=1} a_k (z - z_0)^k \text{ converges}\}$. We will study power series in detail later on, but for now we will just use it to define the exponential function.

Definition 1.16. The **exponential function** $\exp\colon \mathbb{C} \to \mathbb{C}$ is defined via the power series

$$\exp(z) := e^z := \sum_{k=0}^{\infty} \frac{z^k}{k!}. \qquad \blacklozenge$$

Lemma 1.17. *The power series of* $\exp(z)$ *converges absolutely for every* $z \in \mathbb{C}$.

Proof. The limit $\sum_{k=0}^{\infty} \frac{|z|^k}{k!}$ exists and equals $e^{|z|}$ by the definition of the real-valued exponential function. □

Similarly to the exponential function we may define the functions $\cos(z)$ and $\sin(z)$ by extending their known power series to the complex plane.

Definition 1.18. The trigonometric functions $\cos(z)$ and $\sin(z)$ are defined via the power series

$$\cos(z) := \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \quad \text{and} \quad \sin(z) := \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}. \qquad \blacklozenge$$

Theorem 1.19 (Euler's formula)**.** *For any* $z \in \mathbb{C}$ *we have*

$$\exp(iz) = \cos(z) + i\sin(z).$$

In particular when $\theta \in \mathbb{R}$, $\exp(i\theta)$ *lies on the unit circle at angle* θ *counter-clockwise from the positive real axis.*

Proof. We have

$$\begin{aligned}\exp(iz) &= \sum_{k=0}^{\infty} \frac{i^k z^k}{k!} = \sum_{k=0}^{\infty} \frac{i^{2k} z^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{i^{2k+1} z^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{i(-1)^k z^{2k+1}}{(2k+1)!} = \cos(z) + i\sin(z).\end{aligned} \qquad \square$$

The exponential function satisfies the same functional equation as in the real case.

Theorem 1.20. *We have* $e^{z+w} = e^z e^w$ *for all* $z, w \in \mathbb{C}$.

Proof. By the binomial theorem we have

$$e^{z+w} = \sum_{k=0}^{\infty} \frac{(z+w)^k}{k!} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{1}{k!} \binom{k}{l} z^\ell w^{k-\ell}.$$

Since the series converges absolutely, we may change the order of summation and get

$$e^{z+w} = \sum_{\ell=0}^{\infty} \sum_{k=\ell}^{\infty} \frac{z^{\ell} w^{k-\ell}}{\ell!(k-\ell)!} = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \frac{z^{\ell} w^{k}}{\ell! k!} = e^z e^w. \qquad \square$$

1.4 *Polar coordinates and the logarithm*

Recall that **polar coordinates** can be used to specify a point in the plane by giving its radius $r \geq 0$ and angle $\theta \in \mathbb{R}$, in which case its Cartesian coordinates are $(r\cos(\theta), r\sin(\theta))$. By Theorem 1.19 the complex exponential function lets us write this concisely as $re^{i\theta}$.

Conversely, given $z \in \mathbb{C}$ we have $z = |z|e^{i\arg(z)}$, where $\arg(z)$ is the **argument** of z. For $z = 0$ any argument can be used, while for non-zero z the argument is well-defined modulo integer multiples of 2π. A common way to fix the argument is to constrain it to the interval $(-\pi, \pi]$, in which case one can use the specific choice

$$\operatorname{Arg}(x+iy) = \begin{cases} 0, & \text{if } x \geq 0, y = 0; \\ \pi, & \text{if } x < 0, y = 0; \\ \frac{\pi}{2} - \arctan\left(\frac{x}{y}\right), & \text{if } y > 0; \\ -\frac{\pi}{2} - \arctan\left(\frac{x}{y}\right), & \text{if } y < 0. \end{cases}$$

The function Arg is known as the **principal branch** of the argument. It is not continuous along $(-\infty, 0]$, which is called the **branch cut** of the principal branch. See Figure 1.2.

Example 1.21. Let us find the polar representation of $-3+2i$. We first compute the modulus $|-3+2i| = \sqrt{9+4} = \sqrt{13}$. Noting that $y > 0$, we see that its argument is $\frac{\pi}{2} - \arctan\left(\frac{-3}{2}\right) \approx 2.55 \approx 146^\circ$. Hence, $-3+2i = \sqrt{13}e^{i\left(\frac{\pi}{2}+\arctan\left(\frac{3}{2}\right)\right)}$. ◆

Let us next give geometric interpretations to complex arithmetic. The addition $z \mapsto z+a$ is equivalent to addition of vectors of $\mathbb{R}^2$ and just translates z by a. The map $z \mapsto \bar{z}$ can be viewed as a reflection in the real axis. Multiplication is a bit more interesting since it is a combination of rotation and scaling.

Proposition 1.22. *Let* $z, w \in \mathbb{C} \setminus \{0\}$. *Then* zw *satisfies* $|zw| = |z||w|$ *and* $\arg(zw) = \arg(z) + \arg(w) \pmod{2\pi}$. *Similarly,* z/w *satisfies* $|z/w| = |z|/|w|$ *and* $\arg(z/w) = \arg(z) - \arg(w) \pmod{2\pi}$.

Proof. The identity $|zw| = |z||w|$ was part of Lemma 1.9. Let $\arg(zw)$, $\arg(z)$ and $\arg(w)$ denote some choices for the arguments of zw, z and w, respectively.

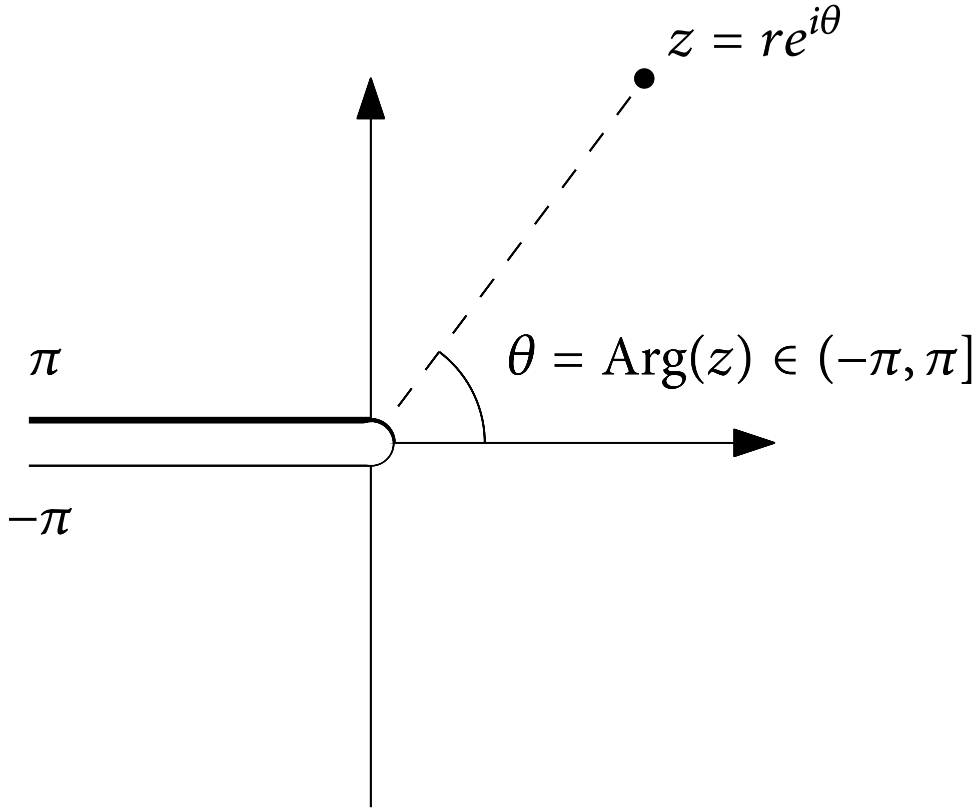

Figure 1.2: Polar coordinates and Arg.

We note that

$$|zw|e^{i\arg(zw)} = zw = |z|e^{i\arg(z)}|w|e^{i\arg(w)} = |zw|e^{i(\arg(z)+\arg(w))},$$

which implies that $\arg(zw) = \arg(z) + \arg(w) + 2\pi ki$ for some $k \in \mathbb{Z}$. The division rule follows from the fact that $w^{-1} = |w|^{-1}e^{-i\arg(w)}$. □

In particular note that multiplying by $re^{i\theta}$ is equivalent to rotation by angle θ and scaling by r.

Example 1.23. Multiplying by $i = e^{i\pi/2}$ is equivalent to rotation by $\pi/2$ radians (90°) counter-clockwise. ◆

Example 1.24. The roots of the polynomial $z^n - 1 = 0$ for $n \geq 1$ are given by $z = e^{2\pi ik/n}$ with $0 \leq k \leq n-1$. They are called the ***n*-th roots of unity**. ◆

In general, a nonzero complex number $z = re^{i\theta}$ has n n-th roots given by $r^{1/n}e^{i\theta/n+2\pi ik/n}$ for $0 \leq k \leq n-1$. When we write $\sqrt[n]{z}$, we can in principle be referring to any of these roots. When a choice has to be singled out, a popular approach is to take $\sqrt[n]{z} = |z|^{1/n}e^{i\operatorname{Arg}(z)/n}$, which is the **principal branch** of the n-th root defined using the principal branch of arg.

Finally, let us note that the map from Cartesian to polar coordinates can be also viewed through logarithms. Indeed, $z = |z|e^{i\arg(z)} = e^{\log(|z|)+i\arg(z)}$ motivates the following definition.

Definition 1.25. Let $z \in \mathbb{C} \setminus \{0\}$. For any choice of $\arg(z)$, the number

$$\log(z) = \log(|z|) + i\arg(z)$$

is called a **logarithm** of z. Here $\log(|z|)$ is the usual real-valued logarithm. The unique choice

$$\operatorname{Log}(z) = \log(|z|) + i\operatorname{Arg}(z)$$

is called the **principal branch** of log. ◆

One has to sometimes be careful with the multivalued nature of arg: We always have $\exp(\log(z)) = \exp(\log(|z|) + i\arg(z)) = |z|e^{i\arg(z)} = z$ for any choice of $\arg(z)$, so taking exponentials of logarithms is generally straightforward. On the other hand $\log(\exp(z))$ can in principle be taken to be any number of the form $z + 2\pi ki$ for $k \in \mathbb{Z}$! For this reason it is often important to keep track of the exact value of $\arg(z)$ being used. Notice in particular that $\operatorname{Log}(\exp(z)) = z$ holds if and only if $\operatorname{Im}(z) \in (-\pi, \pi]$, while for instance $\operatorname{Log}(\exp(2\pi i)) = \operatorname{Log}(1) = 0 \neq 2\pi i$. Similarly, the addition formula $\operatorname{Log}(zw) = \operatorname{Log}(z) + \operatorname{Log}(w)$ holds if and only if $\operatorname{Arg}(z) + \operatorname{Arg}(w) \in (-\pi, \pi]$.

Vector analysis on the complex plane

The purpose of this chapter is to introduce convenient notation and tools for doing calculus on the plane. In particular, we will define complex-valued differential forms and integrals over curves and regions. Our main result in the end will be Stokes' theorem, which is a generalization of the fundamental theorem of calculus. To keep the exposition light, we will restrict ourselves strictly to the plane and the real line.

2.1 *Integrals over curves*

Definition 2.1. A map $\gamma\colon [0,1] \to \mathbb{R}^2$ is called a $\boldsymbol{C^1}$**-curve** if $\gamma'(t)$ exists and is continuous for all $t \in [0,1]$ (with one-sided limits at end points). ◆

Let γ be a C^1-curve. If $f : \gamma([0,1]) \to \mathbb{C}$ is a continuous function defined on the curve, we would like to define integrals such as

$$\int_\gamma f(z)\,dz.$$

Intuitively, we view this as a Riemann sum

$$\int_\gamma f(z)\,dz \approx \sum_{n=1}^{N} f(z_n)(z_{n+1} - z_n),$$

where $(z_n)_{n=1}^N$ is an evenly distributed collection of points on $\gamma([0,1])$, running from $\gamma(0)$ to $\gamma(1)$. We could also do the change of variables where we write $z_n = \gamma(t_n)$ for points $t_1 < \cdots < t_N \in [0,1]$, and get

$$\begin{aligned}\int_\gamma f(z)\,dz &\approx \sum f(\gamma(t_i))(\gamma(t_{i+1}) - \gamma(t_i)) \approx \sum f(\gamma(t_i))\gamma'(t_i)(t_{i+1} - t_i)\\ &\approx \int_0^1 f(\gamma(t))\gamma'(t)\,dt.\end{aligned}$$

In fact, we will soon take the last formula as a definition for the integral. Note that the orientation of γ matters: the integral over the opposite curve $\tilde{\gamma}(t) = \gamma(1-t)$ should satisfy

$$\int_{\tilde{\gamma}} f(z)\,dz = -\int_\gamma f(z)\,dz.$$

The quantity $z_{n+1} - z_n$ should be viewed as an (approximate) tangent vector at the point z_n. The symbol dz in the integral is an example of a *differential* 1-*form*, and it signifies that we should sum up the tangent vectors themselves (multiplied by $f(z_n)$ using complex multiplication).

Another example of a differential 1-form is dx, which maps a tangent vector to its x-component. Thus, when integrating against dx we should intuitively think of the Riemann sum

$$\int_\gamma f(z)\,dx \approx \sum f(z_i)\,\mathrm{Re}(z_{i+1} - z_i).$$

Similarly, we have the 1-form dy, for which

$$\int_\gamma f(z)\,dy \approx \sum f(z_i)\,\mathrm{Im}(z_{i+1} - z_i),$$

and thus we can write $dz = dx + idy$.

Let us now make this more precise.

Definition 2.2. A **tangent vector** on $\mathbb{R}^2$ is a pair (p, v), where $p \in \mathbb{R}^2$ denotes the base point and $v \in \mathbb{R}^2$ the direction. For every fixed $p \in \mathbb{R}^2$ we let $\mathbb{R}^2_p$ be the set of all tangent vectors (p, v) based at p and call it the **tangent space** at p. ◆

The set $\mathbb{R}^2_p$ becomes a vector space isomorphic to $\mathbb{R}^2$ by defining $(p, v) + (p, w) = (p, v + w)$ and $c(p, v) = (p, cv)$ for $c \in \mathbb{R}$. We will also often abuse the notation and identify $(p, v) \in \mathbb{R}^2_p$ with v, as if v already contained the information about its base point.

Definition 2.3. A **(complex-valued) differential 1-form** ω on $U \subset \mathbb{R}^2$ is a map which assigns every $p \in U$ a $\mathbb{R}$-linear map $\omega_p : \mathbb{R}^2_p \to \mathbb{C}$. ◆

By $\mathbb{R}$-linear we mean that $\omega_p(u + v) = \omega_p(u) + \omega_p(v)$ and $\omega_p(cu) = c\omega_p(u)$ for all $u, v \in \mathbb{R}^2_p$ and $c \in \mathbb{R}$.

The standard 1-forms dx and dy simply act on vectors $v = v_1 + v_2 i$ written in the standard basis $(1, i)$ by $dx_p(v) = v_1$ and $dy_p(v) = v_2$. Note that $\mathbb{R}$-linear maps $\mathbb{R}^2_p \to \mathbb{C}$ themselves form a ($\mathbb{C}$-)vector space $(\mathbb{R}^2_p)^*$ with basis dx, dy, and hence one can write any differential 1-form ω in the form

$$\omega = a(p)dx + b(p)dy$$

for some functions $a, b \colon U \to \mathbb{C}$ (we usually drop the subindex p from the forms to lighten the notation). We say that the 1-form ω is C^k for a given $k \geq 0$ if a and b are C^k, meaning that their real and imaginary parts are C^k.

Remark. Why $\mathbb{R}$-linear and not $\mathbb{C}$-linear? We want to have $dx(i) = 0$, but if dx was $\mathbb{C}$-linear, we would have $dx(i) = idx(1) = i$. In other words, dx and

dy would not be independent and things break down. ◆

Definition 2.4. Let γ be a C^1-curve and let ω be a continuous 1-form defined on the curve. The integral of ω over γ is given by

$$\int_\gamma \omega := \int_0^1 \omega_{\gamma(t)}(\gamma'(t))\,dt := \int_0^1 \mathrm{Re}(\omega_{\gamma(t)}(\gamma'(t)))\,dt + i\int_0^1 \mathrm{Im}(\omega_{\gamma(t)}(\gamma'(t)))\,dt,$$

where the integrals on the right are the usual real-valued Riemann or Lebesgue integrals. ◆

Example 2.5. Let $\omega = 2dx + xdy$, and let $\gamma\colon [0,1] \to \mathbb{C}$ be the C^1-curve $\gamma(t) = t + it^2$. Then $\gamma'(t) = 1 + 2ti$, so that $dx(\gamma'(t)) = 1$ and $dy(\gamma'(t)) = 2t$ and

$$\begin{aligned}\int_\gamma \omega = \int_\gamma (2dx + xdy) &= \int_0^1 (2dx(\gamma'(t)) + x(\gamma(t))dy(\gamma'(t)))\,dt \\ &= \int_0^1 (2 + t\cdot 2t)\,dt = 3,\end{aligned}$$

where we viewed x as a function $x(u + vi) = u$. ◆

It is important to note that the integrals do not change if we reparametrize γ (without changing orientation).

Lemma 2.6. *Let γ be a C^1-curve, $\varphi\colon [0,1] \to [0,1]$ an increasing C^1 bijection and $\tilde\gamma = \gamma\circ\varphi$. Then for any continuous 1-form ω we have*

$$\int_\gamma \omega = \int_{\tilde\gamma} \omega.$$

Proof. We have

$$\begin{aligned}\int_\gamma \omega = \int_0^1 \omega_{\gamma(t)}(\gamma'(t))\,dt &= \int_0^1 \omega_{\gamma(\varphi(s))}(\gamma'(\varphi(s)))\varphi'(s)\,ds \\ &= \int_0^1 \omega_{\gamma(\varphi(s))}(\gamma'(\varphi(s))\varphi'(s))\,ds = \int_0^1 \omega_{\tilde\gamma(s)}(\tilde\gamma'(s))\,ds = \int_{\tilde\gamma}\omega,\end{aligned}$$

where we used the change of variables $t = \varphi(s)$ and the linearity of ω. □

As already hinted above, we also define the following complex differential 1-forms.

Definition 2.7. The complex differential 1-forms dz and $d\overline{z}$ are given by

$$dz = dx + idy \quad\text{and}\quad d\overline{z} = dx - idy.$$ ◆

Example 2.8. Let γ be a C^1-curve and $\omega = f\,dz$ a continuous 1-form on $\gamma([0,1])$. Then

$$\int_\gamma \omega = \int_0^1 f(\gamma(t))dz(\gamma'(t))\,dt = \int_0^1 f(\gamma(t))\gamma'(t)\,dt. \qquad \blacklozenge$$

The computations in the examples above might look a bit cumbersome, but we will learn a systematical and more concise way to make them using pull-backs in a bit.

2.2 *Integrals of 2-forms*

We will next define differential 2-forms. Similarly to differential 1-forms which measure signed lengths, differential 2-forms measure signed areas.

Definition 2.9. A **differential 2-form** ω on $U \subset \mathbb{R}^2$ is map which assigns every $p \in U$ a $\mathbb{R}$-bilinear map $\omega_p : (\mathbb{R}^2_p)^2 \to \mathbb{C}$, meaning that

$$\begin{aligned} \omega_p(u+v,w) &= \omega_p(u,w) + \omega_p(v,w) \\ \omega_p(w,u+v) &= \omega_p(w,u) + \omega_p(w,v) \\ \omega_p(cv,w) &= c\omega_p(v,w) = \omega_p(v,cw) \end{aligned}$$

for all $u, v, w \in \mathbb{R}^2_p$ and $c \in \mathbb{R}$. We also require that ω_p is **alternating**, so that

$$\omega_p(u,v) = -\omega_p(v,u)$$

for all $u, v \in \mathbb{R}^2_p$. ◆

The rough idea is that $\omega_p(u,v)$ measures the weighted area of an oriented parallelogram spanned by u and v. The orientation of a parallelogram can be given by assigning an ordering to the edges u and v, i.e. saying which of the two edges comes first. If the second edge is to counter-clockwise direction from the first one, we say that the orientation is positive, and otherwise negative. This is accounted for in the definition by requiring that if we swap u and v then the sign of a 2-form $\omega_p(u,v)$ changes. Notice that this implies that $\omega_p(u,u) = 0$.

Using the multilinearity and alternating property we see that if $u = u_1 + iu_2$ and $v = v_1 + iv_2$ are expressed in the standard basis, then

$$\omega_p(u_1 + iu_2, v_1 + iv_2) = (u_1v_2 - u_2v_1)\omega_p(1,i).$$

Recall that the front factor

$$u_1v_2 - u_2v_1 = \det\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix},$$

is indeed the signed area of the parallelogram spanned by u and v, while the factor $\omega_p(1, i)$ is a constant depending on p but not on u and v. The standard 2-form $dx \wedge dy$ is defined by letting

$$(dx \wedge dy)_p(u, v) = u_1 v_2 - u_2 v_1$$

and satisfies $(dx \wedge dy)_p(1, i) = 1$. Thus, we see that every 2-form ω is of the form $\omega = f(p) dx \wedge dy$ for some complex-valued function $f(p)$.

Above the $\wedge$ in $dx \wedge dy$ stands for the **exterior product** of the 1-forms dx and dy. More generally, if ω_1 and ω_2 are two 1-forms, their exterior product is given by

$$(\omega_1 \wedge \omega_2)(u, v) := \omega_1(u)\omega_2(v) - \omega_2(u)\omega_1(v).$$

One can easily check that this defines an alternating multilinear map and that $(dx \wedge dy)(u, v) = u_1 v_2 - u_2 v_1$ holds.

As for parametrized curves, we have the following analogous definition for integrating over parametrized regions.

Definition 2.10. Let $\varphi\colon [0, 1]^2 \to \mathbb{C}$ be a C^1-map and ω a continuous 2-form defined on $\varphi([0, 1]^2)$. The integral of ω over φ is given by

$$\int_\varphi \omega := \int_0^1 \int_0^1 \omega_{\varphi(x,y)}(\partial_x \varphi(x, y), \partial_y \varphi(x, y))\, dx\, dy,$$

where $\partial_x \varphi(x, y)$ and $\partial_y \varphi(x, y)$ are viewed as tangent vectors in $\mathbb{R}^2_{\varphi(x,y)}$. ◆

Note in particular that if ω is of the form $\omega = f(x, y)\, dx \wedge dy$ for some continuous $f : \varphi([0, 1]^2) \to \mathbb{C}$ and $\varphi = \varphi_1 + i\varphi_2$, then

$$\begin{aligned}\int_\varphi \omega &= \int_0^1 \int_0^1 f(\varphi(x, y))(\partial_x \varphi_1(x, y)\partial_y \varphi_2(x, y) - \partial_x \varphi_2(x, y)\partial_y \varphi_1(x, y))\, dx\, dy \\ &= \int_0^1 \int_0^1 f(\varphi(x, y))\, \mathrm{Jac}_\varphi(x, y)\, dx\, dy,\end{aligned}$$

where Jac_φ is the Jacobian determinant of φ.

As in the case of curves, the value of the integral does not change upon reparametrization.

Lemma 2.11. *Let $\varphi\colon [0, 1]^2 \to \mathbb{C}$ be a C^1-map, $h\colon [0, 1]^2 \to [0, 1]^2$ a C^1 bijection with positive Jacobian determinant and $\tilde{\varphi} = \varphi \circ h$ be a reparametrization of φ. Then for any continuous 2-form ω we have*

$$\int_\varphi \omega = \int_{\tilde{\varphi}} \omega.$$

Proof. We may assume that ω is of the form $f(x, y)\, dx \wedge dy$. Then,

$$\begin{aligned}\int_\varphi \omega &= \int_{[0,1]^2} f(\varphi(x, y))\, \mathrm{Jac}_\varphi(x, y)\, dx\, dy \\ &= \int_{[0,1]^2} f(\varphi(h(u, v)))\, \mathrm{Jac}_\varphi(h(u, v))\, \mathrm{Jac}_h(u, v)\, du\, dv \\ &= \int_{[0,1]^2} f(\tilde{\varphi}(u, v))\, \mathrm{Jac}_{\tilde{\varphi}}(u, v)\, du\, dv \\ &= \int_{\tilde{\varphi}} \omega,\end{aligned}$$

where we used the change-of-variables $(x, y) = h(u, v)$ and chain rule. □

Let us finally note that in the plane it is also possible to directly make sense of integrals of 2-forms instead of looking at parametrized surfaces.

Definition 2.12. Suppose that $f : \mathbb{R}^2 \to \mathbb{C}$ is an integrable function. We then define $\int f(p)\, dx \wedge dy := \int f(p)\, dx\, dy$. ◆

The reason we still care about Definition 2.10 will become clear in a bit when we consider integrals over chains and prove Stokes' theorem.

2.3 *Computing with differential forms and Wirtinger derivatives*

Having defined 1-forms and 2-forms, it is useful to also define a **differential 0-form** on $U \subset \mathbb{R}^2$ simply as a function $f : U \to \mathbb{C}$.

Let us first extend the definition of the exterior product to include 0-forms as well. In general if α and β are n- and m-forms, $\alpha \wedge \beta$ will be an $n+m$-form. In the plane there are no non-zero n-forms for $n \geq 3$, so we only need to consider the cases where $n+m \leq 2$. Since we already defined the product of two 1-forms ω_1 and ω_2 to be given by

$$(\omega_1 \wedge \omega_2)(u, v) = \omega_1(u)\omega_2(v) - \omega_1(v)\omega_2(u) = -(\omega_2 \wedge \omega_1)(u, v),$$

we only need to define the products of 0-forms with either 0-, 1- or 2-forms. In all these cases the definition is the same: If f is a 0-form and ω is an n-form, then

$$(f \wedge \omega)_p(u_1, \dots, u_n) = (\omega \wedge f)_p(u_1, \dots, u_n) = f(p)\omega(u_1, \dots, u_n).$$

In other words, wedging with f just becomes scalar multiplication by $f(p)$. From the definition we see in particular that $\wedge$ distributes over addition and that $\omega \wedge \omega = 0$ for 1-forms ω. For instance,

$$(f dx + g dy) \wedge (u dx + v dy) \wedge p = p(fv - gu) dx \wedge dy$$

for any functions (0-forms) f, g, u, v, p.

The second operation we will define is the **exterior derivative** which maps a k-form to a $k+1$-form. For a differentiable 0-form f it is a 1-form df defined as the directional derivative

$$df_p(v) = \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t},$$

where $(p, v) \in \mathbb{R}^2_p$ is a tangent vector. This means that df_p can also be viewed as the derivative of f in the usual sense as a map $\mathbb{R}^2 \to \mathbb{R}^2$, which is given by the Jacobian matrix. In terms of dx, dy we have

$$df = \partial_x f dx + \partial_y f dy.$$

The derivative of a 1-form $\omega = f dx + g dy$ is defined by taking the derivatives of the 0-forms f and g and wedging with the existing 1-forms, so that

$$d\omega = (df) \wedge dx + (dg) \wedge dy$$

which turns out to equal

$$(\partial_x g - \partial_y f) dx \wedge dy.$$

In two dimensions the derivative of a 2-form is always 0.

An important thing to note is that the coordinates x and y can be viewed as functions on $\mathbb{R}^2$, with $x(u, v) = u$ and $y(u, v) = v$. Hence, dx and dy can be viewed as the exterior derivatives of the functions x and y, which one can check indeed agrees with the definition given earlier. One of the main features of differential forms is that there is nothing special about the coordinate system (x, y), and we can use other coordinates as well. As an example, we could use polar coordinates (r, θ) and the formulas above still work in this new coordinate system. For instance, let us check that $df = \partial_r f \, dr + \partial_\theta f \, d\theta$. We have

$$\begin{aligned} \partial_r f dr + \partial_\theta f d\theta &= \partial_r f(\partial_x r dx + \partial_y r dy) + \partial_\theta f(\partial_x \theta dx + \partial_y \theta dy) \\ &= (\partial_r f \partial_x r + \partial_\theta f \partial_x \theta) dx + (\partial_r f \partial_y r + \partial_\theta f \partial_y \theta) dy \\ &= \partial_x f dx + \partial_y f dy = df, \end{aligned}$$

where we used the chain rule. It is also straightforward to move from one coordinate system to another. For instance, since $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we have $dx = \cos(\theta)dr - r\sin(\theta)d\theta$ and $dy = \sin(\theta)dr + r\cos(\theta)d\theta$, and hence

$$\begin{aligned} dx \wedge dy &= (\cos(\theta)dr - r\sin(\theta)d\theta) \wedge (\sin(\theta)dr + r\cos(\theta)d\theta) \\ &= (\cos^2(\theta) + \sin^2(\theta))rdr \wedge d\theta = rdr \wedge d\theta, \end{aligned}$$

which is the usual area form in polar coordinates.

In fact, the formula $df = \partial_u f du + \partial_v f dv$ for a general coordinate system (u, v) is such a useful mnemonic that it makes sense to look at what happens when we express everything in terms of $dz = dx + idy$ and $d\overline{z} = dx - idy$. We have $dx = \frac{1}{2}(dz + d\overline{z})$ and $dy = \frac{1}{2i}(dz - d\overline{z})$ and thus

$$\begin{aligned} df = \partial_x f dx + \partial_y f dy &= \frac{1}{2}\partial_x f(dz + d\overline{z}) + \frac{1}{2i}\partial_y f(dz - d\overline{z}) \\ &= \frac{1}{2}(\partial_x f - i\partial_y f)dz + \frac{1}{2}(\partial_x f + i\partial_y f)d\overline{z}. \end{aligned}$$

It therefore makes sense to make the following definition.

Definition 2.13. The **complex (or Wirtinger) derivatives** ∂_z and $\partial_{\overline{z}}$ are defined by

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y) \quad \text{and} \quad \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$$ ◆

With these definitions we get the already familiar looking formula

$$df = \partial_z f dz + \partial_{\overline{z}} f d\overline{z}.$$

It is useful to also connect differentials of functions to their differentiability. Recall the following definition from vector analysis.

Definition 2.14. A function $f : U \to \mathbb{C}$ is **(real-)differentiable** (or briefly, $\mathbb{R}$-differentiable) at $z \in U$ if there exists a $\mathbb{R}$-linear functional $df_z : \mathbb{R}^2 \to \mathbb{C}$ such that $f(z + h) = f(z) + df_z(h) + o(|h|)$ as $\mathbb{R}^2 \ni h \to 0$. ◆

Note that if f is differentiable, then the derivative df_z in the definition is unique and agrees with the earlier definition $df_z(v)$ as a directional derivative. In particular, by using the formula above, $f : U \to \mathbb{C}$ is differentiable at z, if and only if it admits a first order Taylor-expansion of the form

$$f(z + h) = f(z) + \partial_z f(z)h + \partial_{\overline{z}} f(z)\overline{h} + o(|h|).$$

Note on terminology: The prefix "real" in "real-differentiability" has been added in order to distinguish the term from "complex-differentiability" that will be defined in next chapter.

The Wirtinger derivatives and the forms dz and $d\overline{z}$ satisfy various useful formulae.

Lemma 2.15. *Let $f: U \to \mathbb{C}$ be a real-differentiable function. Then*

$$\overline{\partial_z f} = \partial_{\overline{z}} \overline{f} \quad \text{and} \quad \overline{\partial_{\overline{z}} f} = \partial_z \overline{f}.$$

Moreover, if $g: V \to U$ is real-differentiable on an open set $V \subset \mathbb{C}$, then the following chain rules hold:

$$\partial_z (f \circ g)(z) = \partial_z f(g(z)) \cdot \partial_z g(z) + \partial_{\overline{z}} f(g(z)) \cdot \partial_z \overline{g}(z),$$
$$\partial_{\overline{z}} (f \circ g)(z) = \partial_z f(g(z)) \cdot \partial_{\overline{z}} g(z) + \partial_{\overline{z}} f(g(z)) \cdot \partial_{\overline{z}} \overline{g}(z).$$

Proof. Exercise. □

The final operation we will look at is the pullback of forms. In the definition below we also consider differential forms on subsets of $\mathbb{R}$ (in the $d = 1$ case). These obey the same rules as forms on subsets of $\mathbb{R}^2$, except that every 1-form is of the form $a\,dx$ for some function a, $da = a'(x)\,dx$ when a is a differentiable 0-form, and there are no non-zero 2-forms.

Definition 2.16. Let $U \subset \mathbb{R}^d$ $(d = 1, 2)$ be open and $f: U \to \mathbb{R}^2$ be a C^1-function. Suppose that ω is a differential m-form defined on $V = f(U)$. Then the pullback of ω is the differential m-form on U given by

$$(f^*\omega)_p(v_1, \ldots, v_m) = \omega_{f(p)}(df_p(v_1), \ldots, df_p(v_m)). \qquad \blacklozenge$$

In the definition df_p is interpreted as mapping the tangent vector v at p to a tangent vector $df_p(v)$ at $f(p)$.

The definition is tailored in such a way that the earlier definitions for integrals over curves and parametrized surfaces can be succinctly stated as

$$\int_\gamma \omega = \int_0^1 \gamma^* \omega \quad \text{and} \quad \int_\varphi \omega = \int_{[0,1]^2} \varphi^* \omega$$

when $\gamma: [0,1] \to \mathbb{C}$ is a C^1-curve or $\varphi: [0,1]^2 \to \mathbb{C}$ is a C^1 parametrized region.

Pullbacks are in fact characterized by the following recursive rules which make computing them easy.

Lemma 2.17. *Let f be a C^1-function (defined either on an open subset of $\mathbb{R}$ or $\mathbb{C}$), and α and β be differential forms. Then the following identities hold:*

- *$f^* g = g \circ f$ when g is a 0-form*
- *$f^*(\alpha + \beta) = f^*\alpha + f^*\beta$*
- *$f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$*

- $f^*(d\alpha) = df^*\alpha$

Proof. Exercise. □

Using the rules we may for instance compute the following useful formulas for pullbacks under $f = u + iv$:

- $f^*dz = df^*(z \mapsto z) = d((z \mapsto z) \circ f) = df = \partial_z f dz + \partial_{\overline{z}} f d\overline{z}$
- $f^*d\overline{z} = df^*(\overline{z}) = d\overline{f} = \partial_z \overline{f} dz + \partial_{\overline{z}} \overline{f} d\overline{z}$
- $f^*(dx \wedge dy) = (\partial_x u \partial_y v - \partial_y u \partial_x v) dx \wedge dy = \mathrm{Jac}_f\, dx \wedge dy$

Example 2.18. We can also leverage the calculus of differential forms to quickly show that $\mathrm{Jac}_f = |\partial_z f|^2 - |\partial_{\overline{z}} f|^2$. Indeed, note that

$$dz \wedge d\overline{z} = (dx + idy) \wedge (dx - idy) = -2idx \wedge dy,$$

giving us the useful formula

$$dx \wedge dy = \frac{i}{2} dz \wedge d\overline{z}.$$

Hence, we have that

$$\begin{aligned}\mathrm{Jac}_f\, dx \wedge dy = f^*(dx \wedge dy) = f^*\left(\frac{i}{2} dz \wedge d\overline{z}\right) = \frac{i}{2}(df \wedge d\overline{f}) \\ = \frac{i}{2}(\partial_z f \partial_{\overline{z}} \overline{f} - \partial_{\overline{z}} f \partial_z \overline{f}) dz \wedge d\overline{z} = (|\partial_z f|^2 - |\partial_{\overline{z}} f|^2) dx \wedge dy,\end{aligned}$$

implying that $\mathrm{Jac}_f = |\partial_z f|^2 - |\partial_{\overline{z}} f|^2$. ◆

2.4 *Chains and Stokes' theorem*

Our goal in this section is to prove Stokes' theorem, which roughly speaking says that $\int_S d\omega = \int_{\partial S} \omega$, when ω is a 1-form, S is a suitable region in the plane and ∂S is its boundary.

The regions we will cover are ones that can be built by using images of finitely many squares (fancily called 2-cells).

Definition 2.19. A **(C^1 singular) n-cell** ($n = 1, 2$) is a C^1 map $I\colon [0,1]^n \to \mathbb{R}^2$, while a 0-cell is a singleton map $I\colon \{0\} \to \mathbb{R}^2$. ◆

Thus, 0-cells can be viewed as points, 1-cells as parametrized curves and 2-cells as parametrized regions. The integral of an n-form over an n-cell for $n = 1, 2$ is defined by $\int_I \omega = \int_{[0,1]^n} I^*\omega$. For a 0-cell I we simply set $\int_I \omega := \omega(I(0))$.

Definition 2.20. A (***C^1* singular) *n*-chain** C is a finite formal linear combination of n-cells $I_1, \dots, I_m$, i.e. $C = \sum_{k=1}^m c_k I_k$ for some coefficients $c_k \in \mathbb{C}$. ◆

Note that the sum $C = \sum_{k=1}^m c_k I_k$ is completely formal and one should view I_k as symbolic variables. (In particular, it is *not* the pointwise sum of functions.) Typically, the coefficients c_k will be integers. If C is an n-chain, we define

$$\int_C \omega = \sum_{k=1}^m c_k \int_{I_k} \omega = \sum_{k=1}^m c_k \int_{[0,1]^n} I_k^* \omega$$

for any continuous n-form ω. We also denote the empty chain C by 0 and define the integral over an empty chain as 0 as well.

The **boundary** of a 1-cell I is a 0-chain ∂I given by

$$\partial I = I|_1 - I|_0,$$

where $I|_a \colon \{0\} \to \mathbb{C}$ for $a \in \{0, 1\}$ is the 0-cell mapping $I|_a(0) = I(a)$. Similarly, the boundary of a 2-cell S is a 1-chain ∂S given by

$$\partial S = S|_{[0,1]} + S|_{[1,1+i]} - S|_{[i,1+i]} - S|_{[0,i]},$$

where $S|_{[a,b]} \colon [0, 1] \to \mathbb{C}$ is the 1-cell mapping

$$S|_{[a,b]}(t) = S((1 - t)a + tb).$$

Notice that the boundary of a 2-cell is oriented counter-clockwise. The boundary operation extends linearly to n-chains.

Definition 2.21. Let C_1 and C_2 be two n-chains. We say that C_1 and C_2 are equivalent if $\int_{C_1} \omega = \int_{C_2} \omega$ for every continuous n-form ω. ◆

In particular, a constant map $[0, 1]^n \to \mathbb{C}$ for $n \geq 1$ is an n-chain consisting of a single cell and equivalent to an empty chain. Note also that if we reparametrize one of the cells I in the chain by replacing it with $I \circ \varphi$ for some C^1 bijection $\varphi \colon [0, 1]^n \to [0, 1]^n$ with positive Jacobian determinant, we get an equivalent chain by Lemma 2.6 and Lemma 2.11.

Let us look at some important special cases. The closure of a disc $B(z_0, r)$ can be written as a 2-cell S via the map $S(x, y) = z_0 + xre^{2\pi i y}$. Note that $S|_{[0,1]} = S|_{[i,i+1]}$ and that $S|_{[0,i]} \equiv z_0$ is a constant map, so that ∂S is equivalent to the 1-cell $I(t) = S|_{[1,1+i]} = z_0 + re^{2\pi i t}$, which parametrizes the circle $\partial B(z_0, r)$. We will abuse the notation and occasionally refer to the 2-chain S by writing $B(z_0, r)$, so that for instance

$$\int_{B(z_0,r)} \omega = \int_S \omega$$

and

$$\int_{\partial B(z_0,r)} \omega = \int_I \omega.$$

More generally, an annulus $A(z_0, r, R) = \{z \in \mathbb{C} : r < |z - z_0| < R\}$ for $0 < r < R$ can be written as a 2-cell S via the map $S(x, y) = z_0 + ((1-x)r + xR)e^{2\pi i y}$ and $\partial A(z_0, r, R)$ is equivalent to $\partial B(z_0, R) - \partial B(z_0, r)$, when both are interpreted as 1-chains.

The main theorem of this chapter is the following.

Theorem 2.22 (Stokes' theorem). *Let C be an n-chain ($n = 1, 2$) and ω a real-differentiable $n-1$-form such that $d\omega$ is a continuous n-form. Then*

$$\int_C d\omega = \int_{\partial C} \omega.$$

Proof. It is enough to show this in the case where $C = I$, where I is an n-cell for $n = 1, 2$. Note that in this case we have

$$\int_C d\omega = \int_I d\omega = \int_{[0,1]^n} I^*(d\omega) = \int_{[0,1]^n} dI^*\omega = \int_{[0,1]^n} d\alpha,$$

where $\alpha = I^*\omega$ is an $n-1$-form on $[0,1]^n$.

Case $n = 1$: In this case α is just a C^1-function, and we get

$$\int_{[0,1]} d\alpha = \int_0^1 \alpha'(x)\,dx = \alpha(1) - \alpha(0) = \omega(I(1)) - \omega(I(0)) = \int_{\partial I} \omega$$

by the fundamental theorem of calculus.

Case $n = 2$: In this case we have $\alpha = a\,dx + b\,dy$ for some differentiable functions a, b and $d\alpha = (\partial_x b - \partial_y a)dx \wedge dy$, where $\partial_x b - \partial_y a$ is continuous by assumption. Note however that we do *not* know whether $\partial_x b$ and $\partial_y a$ are continuous on their own, which means that we cannot simply split the integral as a sum

$$\int_{[0,1]^2} d\alpha = \int_0^1 \int_0^1 (-\partial_y a)\,dy\,dx + \int_0^1 \int_0^1 (\partial_x b)\,dx\,dy$$

and use the one-dimensional fundamental theorem of calculus in the inner integrals to conclude. Instead, we are going to use an argument that is usually used to prove a related result known as Goursat's theorem.

Our goal is equivalent to showing that

$$\left| \int_{[0,1]^2} d\alpha - \int_{\partial[0,1]^2} \alpha \right| = 0,$$

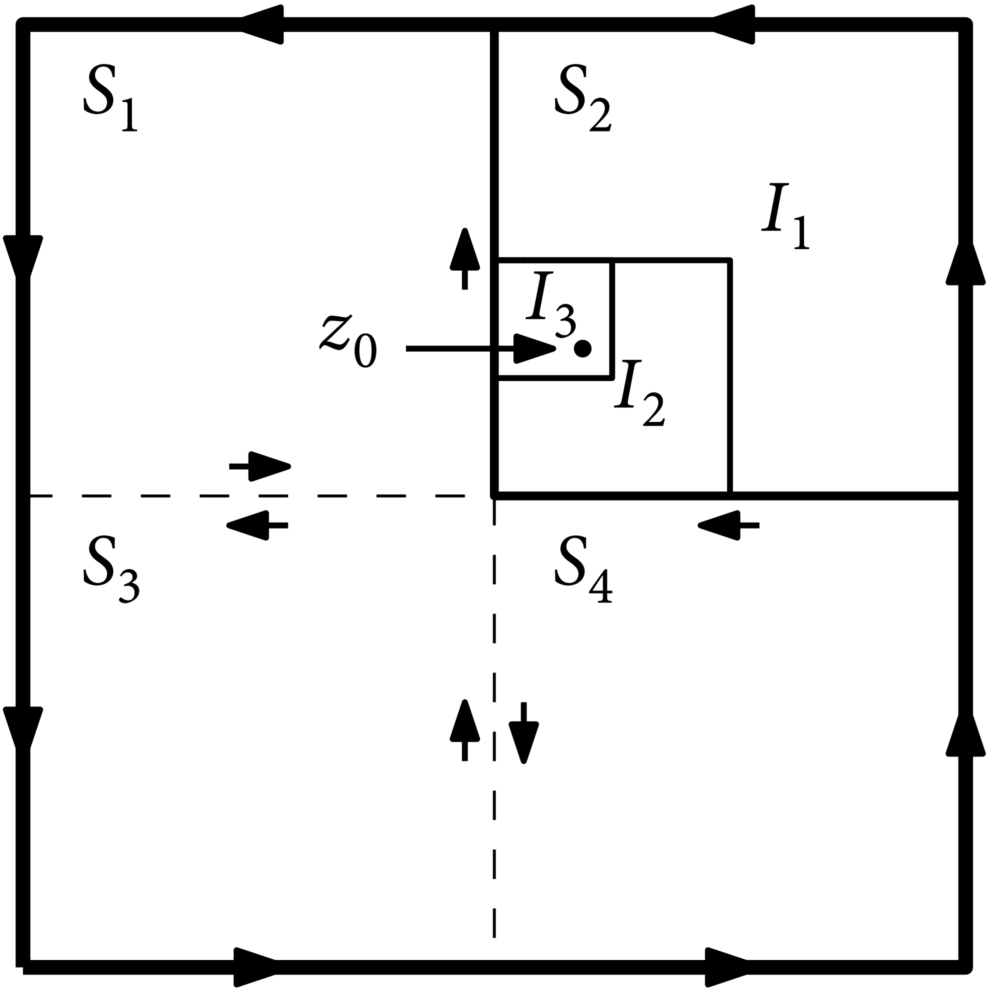

Figure 2.1: Subdivision of $[0,1]^2$.

and we will do this by bounding the left-hand side in a clever way. Let us split $[0,1]^2$ into four sub-squares $S_1, \ldots, S_4$ of side length $1/2$ (see Figure 2.1). Each S_j is a 2-cell on its own (via a map $[0,1]^2 \to S_j$ which scales by $1/2$ and translates by 0 or $1/2$ units in either coordinate) and we can write

$$\int_{[0,1]^2} d\alpha = \sum_{j=1}^{4} \int_{S_j} d\alpha.$$

Notice also that the boundaries of S_j which are inside $[0,1]^2$ cancel out each other and hence also

$$\int_{\partial[0,1]^2} \alpha = \sum_{j=1}^{4} \int_{\partial S_j} \alpha.$$

Let now $I_1 = S_j$ be a sub-square for which

$$\left| \int_{S_j} d\alpha - \int_{\partial S_j} \alpha \right|$$

is maximal. By the triangle inequality we then get the bound

$$\left| \int_{[0,1]^2} d\alpha - \int_{\partial[0,1]^2} \alpha \right| \leq 4 \left| \int_{I_1} d\alpha - \int_{\partial I_1} \alpha \right|.$$

We can now continue subdividing inductively. Suppose that we have already defined I_k and showed the bound

$$\left| \int_{[0,1]^2} d\alpha - \int_{\partial[0,1]^2} \alpha \right| \leq 4^k \left| \int_{I_k} d\alpha - \int_{\partial I_k} \alpha \right|.$$

We can then again divide I_k into four sub-squares $S_j^{(k)}$ and choose I_{k+1} to be a one for which

$$\left| \int_{S_j^{(k)}} d\alpha - \int_{\partial S_j^{(k)}} \alpha \right|$$

is maximal. We then get the bound

$$\left| \int_{[0,1]^2} d\alpha - \int_{\partial[0,1]^2} \alpha \right| \leq 4^k \left| \int_{I_k} d\alpha - \int_{\partial I_k} \alpha \right| \leq 4^{k+1} \left| \int_{I_{k+1}} d\alpha - \int_{\partial I_{k+1}} \alpha \right|.$$

This process gives us a decreasing sequence $[0,1]^2 \supset I_1 \supset I_2 \supset \dots$ of closed squares I_k with side-length 2^{-k}, whose intersection is a single point $z_0 = x_0 + iy_0 \in [0,1]^2$. We will next use the real-differentiability of α at z_0 to conclude. Let us write $I_k = [x_{k,0}, x_{k,1}] \times [y_{k,0}, y_{k,1}]$, use $z = x + iy$ as our variable of integration, and note that for any $k \geq 1$ we have

$$\begin{aligned}
\int_{\partial I_k} \alpha &= \int_{\partial I_k} (a\,dx + b\,dy) \\
&= \int_{\partial I_k} (a(z_0) + \partial_x a(z_0)(x - x_0) + \partial_y a(z_0)(y - y_0) + o_{z\to z_0}(|z - z_0|))\,dx \\
&+ \int_{\partial I_k} (b(z_0) + \partial_x b(z_0)(x - x_0) + \partial_y b(z_0)(y - y_0) + o_{z\to z_0}(|z - z_0|))\,dy \\
&= 2^{-k}\partial_y a(z_0)(y_{k,0} - y_{k,1}) + 2^{-k}\partial_x b(z_0)(x_{k,1} - x_{k,0}) + o_{k\to\infty}(4^{-k}) \\
&= 4^{-k}(\partial_x b(z_0) - \partial_y a(z_0)) + o_{k\to\infty}(4^{-k}),
\end{aligned}$$

since $\int_{\partial I_k} f(x)\,dx = \int_{\partial I_k} f(y)\,dy = 0$ for any continuous function $f : \mathbb{R} \to \mathbb{C}$.

We can also compute

$$\begin{aligned}\int_{I_k} d\alpha &= \int_{I_k} (\partial_x b(z) - \partial_y a(z))\, dx\, dy \\ &= \int_{I_k} (\partial_x b(z_0) - \partial_y a(z_0) + o_{z\to z_0}(1))\, dx\, dy \\ &= 4^{-k}(\partial_x b(z_0) - \partial_y a(z_0)) + o_{k\to\infty}(4^{-k}).\end{aligned}$$

Therefore,

$$\left|\int_{[0,1]^2} d\alpha - \int_{\partial[0,1]^2} \alpha\right| \leq 4^k \left|\int_{I_k} d\alpha - \int_{\partial I_k} \alpha\right| \leq 4^k \cdot o_{k\to\infty}(4^{-k}) \to 0$$

as $k \to \infty$. □

Holomorphic functions

3.1 *Complex differentiability*

From now on, $U \subset \mathbb{C}$ will always be an open set.

Definition 3.1. A function $f: U \to \mathbb{C}$ is **complex-differentiable** (or briefly, **$\mathbb{C}$-differentiable**) at $z \in U$ if the limit

$$f'(z) \coloneqq \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. We call $f'(z)$ the derivative of f at $z \in \mathbb{C}$. ◆

In the definition above it is crucial that $h \in \mathbb{C} \setminus \{0\}$ is allowed to approach 0 in any manner! Formally, $f'(z) = w$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $h \in \mathbb{C}$ with $0 < |h| < \delta$ we have $\left|\frac{f(z+h)-f(z)}{h} - w\right| < \varepsilon$.

Note on terminology: If we say that a map is C^1, we mean that its partial derivatives exist and are continuous, which in turn implies $\mathbb{R}$-differentiability but not necessarily $\mathbb{C}$-differentiability. The notation f' is reserved for limits of difference quotients of the form $(f(z+h) - f(z))/h$, and applies to functions $\mathbb{C} \to \mathbb{C}$ (where $h \in \mathbb{C}$) or $\mathbb{R} \to \mathbb{C}$ (where $h \in \mathbb{R}$), where the latter case is mainly used for curves. In other cases we will use the partial derivative notation ∂_x etc., or the total derivative df_z at z, which is a $\mathbb{R}$-linear map $df_z : \mathbb{C} \to \mathbb{C}$ as in Definition 2.14 and also coincides with the exterior derivative of f.

Example 3.2. (i) The function $f(z) = z$ is everywhere $\mathbb{C}$-differentiable with $f'(z) = 1$ since $\frac{(z+h)-z}{h} = 1$ for all $h \in \mathbb{C}$.

(ii) An important counter-example is the function $f(z) = \overline{z}$ for which

$$\frac{\overline{(z+h)} - \overline{z}}{h} = \frac{\overline{h}}{h}$$

does not have a limit as $h \to 0$.

(iii) The difference quotient of the function $|z|^2$ is

$$\frac{|z+h|^2 - |z|^2}{h} = \frac{|z|^2 + |h|^2 + z\overline{h} + \overline{z}h - |z|^2}{h} = \frac{|h|^2}{h} + z\frac{\overline{h}}{h} + \overline{z},$$

which has a limit as $h \to 0$ if and only if $z = 0$. ◆

Let us begin by connecting complex differentiability to the Wirtinger derivatives defined earlier.

Theorem 3.3. *Suppose that $f : U \to \mathbb{C}$ is $\mathbb{C}$-differentiable at $z \in U$. Then f is $\mathbb{R}$-differentiable at z, $f'(z) = \partial_z f(z)$ and $\partial_{\overline{z}} f(z) = 0$. Conversely, if f is $\mathbb{R}$-differentiable at z and $\partial_{\overline{z}} f(z) = 0$, then f is $\mathbb{C}$-differentiable at z.*

Proof. Suppose first that f is $\mathbb{C}$-differentiable at z. Then we have

$$\lim_{h\to 0} \frac{f(z+h) - f(z) - f'(z)h}{h} = 0,$$

meaning that

$$f(z+h) - f(z) - f'(z)h = o(|h|)$$

as $h \to 0$. Thus, the expansion in Definition 2.14 holds with $df_z(h) = f'(z)h$ and f is $\mathbb{R}$-differentiable at z. Since $df_z(h) = \partial_z f(z)h + \partial_{\overline{z}} f(z)\overline{h} = f'(z)h$ has to hold for every $h \in \mathbb{C}$, we see by equating the coefficients of h and $\overline{h}$ that $\partial_z f(z) = f'(z)$ and $\partial_{\overline{z}} f(z) = 0$.

Conversely, if f is $\mathbb{R}$-differentiable at z and $\partial_{\overline{z}} f(z) = 0$, the expansion

$$f(z+h) = f(z) + \partial_z f(z)h + \partial_{\overline{z}} f(z)\overline{h} + o(|h|) = f(z) + \partial_z f(z)h + o(|h|)$$

holds and hence

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \partial_z f(z). \qquad \square$$

Remark. In particular, one can check that f is $\mathbb{C}$-differentiable at z if and only if we have the expansion

$$f(z+h) = f(z) + df_z(h) + o(|h|),$$

with $\mathbb{C}$-linear df_z, which also fits the term '$\mathbb{C}$-differentiable', since in ordinary ($\mathbb{R}$-)differentiability we just required df_z to be $\mathbb{R}$-linear. ◆

Derivatives of complex functions satisfy many familiar calculus rules.

Theorem 3.4. *Let f and g be $\mathbb{C}$-differentiable at z. Then*

- *$f + g$ is $\mathbb{C}$-differentiable at z with*

$$(f+g)'(z) = f'(z) + g'(z),$$

- *$f \cdot g$ is $\mathbb{C}$-differentiable at z with*

$$(f \cdot g)'(z) = f'(z)g(z) + g'(z)f(z), \text{ and}$$

- *if* $g(z) \neq 0$*, then* $\frac{f}{g}$ *is* $\mathbb{C}$*-differentiable at* z *with*

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - g'(z)f(z)}{g(z)^2}.$$

Moreover, if f *is* $\mathbb{C}$*-differentiable at* w*,* g *is* $\mathbb{C}$*-differentiable at* z *and* $g(z) = w$*, then* $f \circ g$ *is* $\mathbb{C}$*-differentiable at* z *with*

$$(f \circ g)'(z) = f'(g(z))g'(z) = f'(w)g'(z).$$

Proof. Let us leave the quotient rule as an exercise.

One can prove the rules directly from Definition 3.1 basically word-by-word as they are shown for functions $\mathbb{R} \to \mathbb{R}$ in basic analysis courses.

For instance, the sum rule is straightforward, $f + g$ is $\mathbb{C}$-differentiable since

$$\frac{(f+g)(z+h) - (f+g)(z)}{h} = \frac{f(z+h) - f(z)}{h} + \frac{g(z+h) - g(z)}{h},$$

which tends to $f'(z) + g'(z)$ as $h \to 0$. Similarly, for products, we may write

$$\begin{aligned} &\frac{f(z+h-)g(z+h) - f(z)g(z)}{h} \\ &= \frac{(f(z+h) - f(z))g(z+h) + (g(z+h) - g(z))f(z)}{h}, \end{aligned}$$

which tends to $f'(z)g(z) + g'(z)f(z)$.

For the chain rule, we could again mimic the proof for real-valued functions, but let us use the general chain-rule for $\mathbb{R}$-differentiable functions instead. Since f and g are $\mathbb{C}$-differentiable, they are also $\mathbb{R}$-differentiable, and We have

$$d(f \circ g)_z(h) = (df_{g(z)} \circ dg_z)(h).$$

By $\mathbb{C}$-differentiability, $dg_z(h) = g'(z)h$ and $df_{g(z)}(h) = f'(g(z))h$ for all $h \in \mathbb{C}$, and hence

$$d(f \circ g)_z(h) = df_{g(z)}(g'(z)h) = f'(g(z))g'(z)h.$$

Thus, $d(f \circ g)_z$ is $\mathbb{C}$-linear and hence $f \circ g$ is $\mathbb{C}$-differentiable with derivative $f'(g(z))g'(z)$. □

Complex-differentiability at a single point is still a fairly weak condition, but if a function is complex-differentiable in an open set, it becomes actually a very strong condition with profound consequences as we shall see later on.

Definition 3.5. A function $f : U \to \mathbb{C}$ is called **holomorphic** in U if it is $\mathbb{C}$-differentiable at every $z \in U$. ◆

Example 3.6. From Theorem 3.4 it easily follows that polynomials $f(z) = a_n z^n + \cdots + a_1 z + a_0$, $a_0, \ldots, a_n \in \mathbb{C}$, are holomorphic on the whole complex plane. Rational functions $f(z)/g(z)$ where f and g are polynomials are also holomorphic in the open set $\{z \in \mathbb{C} : g(z) \neq 0\}$. ◆

Theorem 3.7. *The functions* exp, cos *and* sin *are holomorphic in* $\mathbb{C}$ *and* $\exp' = \exp$, $\cos' = -\sin$, $\sin' = \cos$.

Proof. Let us start with exp. Note that by Theorem 1.20 and Theorem 1.19 we have that

$$\exp(z) = \exp(x + iy) = \exp(x)\exp(iy) = \exp(x)(\cos(y) + i\sin(y))$$

for every $x, y \in \mathbb{R}$, $z = x + iy$. In particular, note that $\exp(z)$ is C^1, since $\exp, \cos, \sin : \mathbb{R} \to \mathbb{R}$ are. Thus,

$$\partial_x \exp(z) = \exp(x)(\cos(y) + i\sin(y)) = \exp(z)$$

and

$$\partial_y \exp(z) = \exp(x)(-\sin(y) + i\cos(y)) = i\exp(z).$$

Since exp is C^1, it is $\mathbb{R}$-differentiable, and we moreover have $\partial_z \exp(z) = \frac{1}{2}(\partial_x - i\partial_y)\exp(z) = \exp(z)$ and $\partial_{\overline{z}} \exp(z) = \frac{1}{2}(\partial_x + i\partial_y)\exp(z) = 0$, implying that exp is $\mathbb{C}$-differentiable and $\exp' = \exp$.

The functions cos and sin can be handled by noting that $\cos(z) = (\exp(iz) + \exp(-iz))/2$ and $\sin(z) = (\exp(iz) - \exp(-iz))/(2i)$. □

Theorem 3.8. *The principal branch of the logarithm* Log *is holomorphic in* $\mathbb{C} \setminus (-\infty, 0]$ *and* $\mathrm{Log}'(z) = 1/z$.

Proof. Recall that $\mathrm{Log}(re^{i\theta}) = \log(r) + i\theta$ when $\theta \in (-\pi, \pi)$. Thus, Log yields a bijection between $\mathbb{C} \setminus (-\infty, 0]$ and the strip $\{z \in \mathbb{C} : \mathrm{Im}(z) \in (-\pi, \pi)\}$, and we have $(\exp \circ \mathrm{Log})(z) = z$ for $z \in \mathbb{C} \setminus (-\infty, 0]$. Since exp is differentiable with $\exp'(z) = \exp(z) \neq 0$, we have by the inverse function theorem that Log is $\mathbb{R}$-differentiable and thus by the chain rule for any $h \in \mathbb{C}$,

$$h = d(\exp \circ \mathrm{Log})_z(h) = \exp(\mathrm{Log}(z))d\mathrm{Log}_z(h) = zd\mathrm{Log}_z(h),$$

so that $d\mathrm{Log}_z(h) = h/z$, which implies that $\partial_z \mathrm{Log}(z) = 1/z$ and $\partial_{\overline{z}} \mathrm{Log}(z) = 0$. □

Let us end this section by studying complex-differentiability in the (x, y) coordinates. Suppose that f is $\mathbb{R}$-differentiable at z and let $u = \mathrm{Re}(f)$ and

$v = \mathrm{Im}(f)$. The condition $\partial_{\overline{z}} f(z) = 0$ is equivalent with

$$\begin{aligned} 0 &= \frac{1}{2}(\partial_x u + i\partial_y u) + \frac{i}{2}(\partial_x v + i\partial_y v) \\ &= \frac{1}{2}(\partial_x u - \partial_y v) + \frac{i}{2}(\partial_y u + \partial_x v). \end{aligned}$$

Looking separately at the real and imaginary parts of this equation gives us the so-called Cauchy–Riemann equations.

Definition 3.9. Functions $u, v\colon U \to \mathbb{R}$ satisfy the **Cauchy–Riemann equations** at the point $z = (x, y) \in U$ if

$$\partial_x u(x, y) = \partial_y v(x, y) \quad \text{and} \quad \partial_y u(x, y) = -\partial_x v(x, y).$$ ◆

We can also go from Cauchy–Riemann equations back to complex differentiability.

Theorem 3.10. *If $u, v\colon U \to \mathbb{R}$ are $\mathbb{R}$-differentiable and satisfy the Cauchy–Riemann equations at $z \in U$, then $f = u + iv$ is $\mathbb{C}$-differentiable at z.*

Proof. As u and v are $\mathbb{R}$-differentiable at z, so is $f = u + iv$. Since u and v satisfy the Cauchy–Riemann equations we have $\partial_{\overline{z}} f(z) = 0$ and the claim follows from Theorem 3.3. □

Cauchy–Riemann equations can be used to solve u given v or v given u.

Example 3.11. Suppose that $f = u + iv$ is a holomorphic function in $\mathbb{C}$ whose real part u equals

$$u(x, y) = x^2 + 2x + 1 - y^2.$$

What is the imaginary part of f? To find v, we can solve the Cauchy–Riemann equations

$$\begin{aligned} \partial_y v(x, y) &= \partial_x u(x, y) = 2x + 2 \\ \partial_x v(x, y) &= -\partial_y u(x, y) = 2y. \end{aligned}$$

Integrating the first equation with respect to y tells us that

$$v(x, y) = 2xy + 2y + C(x),$$

where $C(x)$ is some constant of integration depending on x. Similarly, integrating the second equation with respect to x gives us

$$v(x, y) = 2xy + D(y)$$

where $D(y)$ is some constant depending on y. Combining the two equations, we must have

$$2xy + 2y + C(x) = 2xy + D(y),$$

which implies that $C(x) = D(y) - 2y$. Since the right-hand side does not depend on x, $C(x) = C$ has to be a (real) constant. Thus,

$$v(x, y) = 2xy + 2y + C,$$

and in total, we have

$$f(x, y) = x^2 + 2x + 1 - y^2 + i(2xy + 2y + C)$$

for some arbitrary constant $C \in \mathbb{R}$. We can also write this in terms of z by noting that $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/(2i)$, which we can substitute to get

$$\begin{aligned} f(z) &= \left(\frac{z+\overline{z}}{2}\right)^2 + 2\cdot\frac{z+\overline{z}}{2} + 1 - \left(\frac{z-\overline{z}}{2i}\right)^2 \\ &\quad + i\left(2\cdot\frac{z+\overline{z}}{2}\cdot\frac{z-\overline{z}}{2i} + 2\cdot\frac{z-\overline{z}}{2i} + C\right) \\ &= \frac{z^2+\overline{z}^2+2|z|^2}{4} + z + \overline{z} + 1 + \frac{z^2+\overline{z}^2-2|z|^2}{4} + \frac{z^2-\overline{z}^2}{2} + z - \overline{z} + iC \\ &= z^2 + 2z + 1 + iC = (1+z)^2 + iC. \end{aligned}$$

◆

3.2 *Contour integrals of holomorphic functions*

Definition 3.12. A **contour** in U is a 1-chain $\gamma = \gamma_1 + \cdots + \gamma_n$ consisting of finitely many C^1-curves $\gamma_k\colon [0,1] \to U$ such that $\gamma_k(1) = \gamma_{k+1}(0)$ for all $1 \le k \le n-1$. We also define the piecewise C^1 parametrization $\gamma(t)$ for $t \in [0,1]$ by setting

$$\gamma(t) := \gamma_k(t - (k-1)/n)$$

for $1 \le k \le n$ and $t \in [(k-1)/n, k/n]$. A contour γ with $\gamma(0) = \gamma(1)$ is called **closed**. ◆

Our goal in this section is to consider integrals of the form

$$\int_\gamma f(z)\,dz$$

for a holomorphic function f and contour γ. We begin with the following analogue of the fundamental theorem of calculus.

Theorem 3.13. *Let γ be a contour in U and let f be a holomorphic function in*

U such that f' is continuous[1]*. Then we have*

$$\int_\gamma f'(z)\,dz = f(\gamma(1)) - f(\gamma(0)).$$

Proof. Since f is holomorphic, we have $\partial_{\overline{z}} f(z) = 0$ for $z \in U$, and hence $df = \partial_z f(z)\,dz + \partial_{\overline{z}} f(z)\,d\overline{z} = f'(z)\,dz$. By Stokes' theorem we get

$$\int_\gamma f'(z)\,dz = \int_\gamma df = \int_{\partial\gamma} f = f(\gamma(1)) - f(\gamma(0)). \qquad \square$$

Let us next compute an important example to get some hands-on feeling for contour integrals.

Lemma 3.14. *For any $z_0 \in \mathbb{C}$, $n \in \mathbb{Z}$ and $r > 0$ we have*

$$\int_{\partial B(z_0,r)} (z - z_0)^n\,dz = \begin{cases} 2\pi i, & \textit{if } n = -1 \\ 0, & \textit{otherwise} \end{cases}$$

Proof. By the change of variables $w = z - z_0$ we have $dw = dz$ and

$$\int_{\partial B(z_0,r)} (z - z_0)^n\,dz = \int_{\partial B(0,r)} z^n\,dz.$$

If $n \neq -1$ then z^n has the antiderivative $\frac{z^{n+1}}{n+1}$ which is holomorphic in the punctured plane $\mathbb{C} \setminus \{0\}$ (or even $\mathbb{C}$ in the case $n \geq 0$, but we don't need this). Since the contour $\gamma(t) = re^{2\pi i t}$ lies inside $\mathbb{C} \setminus \{0\}$, we see that

$$\int_{\partial B(0,r)} z^n\,dz = \frac{(\gamma(1))^{n+1}}{n+1} - \frac{(\gamma(0))^{n+1}}{n+1} = 0.$$

If $n = -1$, then z^{-1} has locally antiderivative $\log(z)$, but it cannot be defined as a single-valued function in $\mathbb{C} \setminus \{0\}$: for instance the principal branch Log is defined only on $\mathbb{C} \setminus (-\infty, 0]$, and since $\partial B(0,r)$ crosses the negative real axis the argument we used above for $n \neq -1$ fails (as it should, since the answer in this case is not 0). We will therefore compute the integral directly. Note that $\gamma'(t) = 2\pi i r e^{2\pi i t}$. We thus have

$$\int_\gamma z^{-1}\,dz = \int_0^1 \frac{\gamma'(t)}{\gamma(t)}\,dt = \int_0^1 \frac{2\pi i r e^{2\pi i t}}{re^{2\pi i t}}\,dt = 2\pi i. \qquad \square$$

Remark. We *could* have also proven the $n = -1$ case by using Log as an an-

[1] Assuming continuity of f' is in fact redundant, since we will later on see that holomorphic functions are smooth.

tiderivative and integrating along $\gamma(t) = re^{2\pi i(1-\varepsilon)(t-1/2)}$ ($t \in [0,1]$) for some small $\varepsilon > 0$. Notice that we have cut out a small arc of the circle so that the resulting curve does not cross the negative real axis. The integral then equals

$$\int_\gamma z^{-1}\,dz = \mathrm{Log}(re^{\pi i(1-\varepsilon)}) - \mathrm{Log}(re^{-\pi i(1-\varepsilon)}) = 2\pi i(1-\varepsilon)$$

and tends to $2\pi i$ as $\varepsilon \to 0$. ◆

Above we saw that the value of the integral $\int_\gamma f'(z)\,dz$ only depends on the endpoints of γ. Here we assumed that we are integrating the derivative of a holomorphic function, but our main goal in this section is to show that also in general $\int_\gamma f(z)\,dz$ does not change if we move γ in a continuous manner while keeping the endpoints of γ fixed. The technical term for such a deformation is 'homotopy'.

Definition 3.15. Let α and β be two contours in U with $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. Then α and β are **homotopic** in U if there exists a continuous function $H\colon [0,1]^2 \to U$ such that

$$\begin{aligned} H(\cdot,0) &= \alpha(\cdot) \\ H(\cdot,1) &= \beta(\cdot) \\ H(0,\cdot) &= \alpha(0) = \beta(0) \\ H(1,\cdot) &= \alpha(1) = \beta(1). \end{aligned}$$

◆

Theorem 3.16. *Let U be a convex open set and let α and β be two contours in U with common endpoints. Then α and β are homotopic in U.*

Proof. We may define the linear interpolation $H(x,t) = (1-t)\alpha(x) + t\beta(x)$ for $x, t \in [0,1]$ between the two contours. It is clearly continuous and fixes the endpoints. By convexity $H([0,1]^2) \subset U$. □

The proof of the following theorem essentially says that one can from the existence of a continuous homotopy of contours also deduce the existence of a nice piecewise C^1 homotopy. For our purposes it is convenient to state the theorem in terms of chains.

Theorem 3.17. *Let α and β be two contours in U with common endpoints and homotopic to each other in U. Then there exists a 2-chain in U whose boundary is equivalent to the 1-chain $\alpha - \beta$.*

Proof. Let H be a homotopy between α and β in U. Suppose that α consists of n arcs (1-cells) and β of m arcs, so that $t \mapsto \alpha(t)$ is C^1 when restricted to a subinterval of the form $[(j-1)/n, j/n]$ for $1 \le j \le n$ and $t \mapsto \beta(t)$ is C^1 when restricted to $[(j-1)/m, j/m]$ for $1 \le j \le m$. By uniform continuity, we can

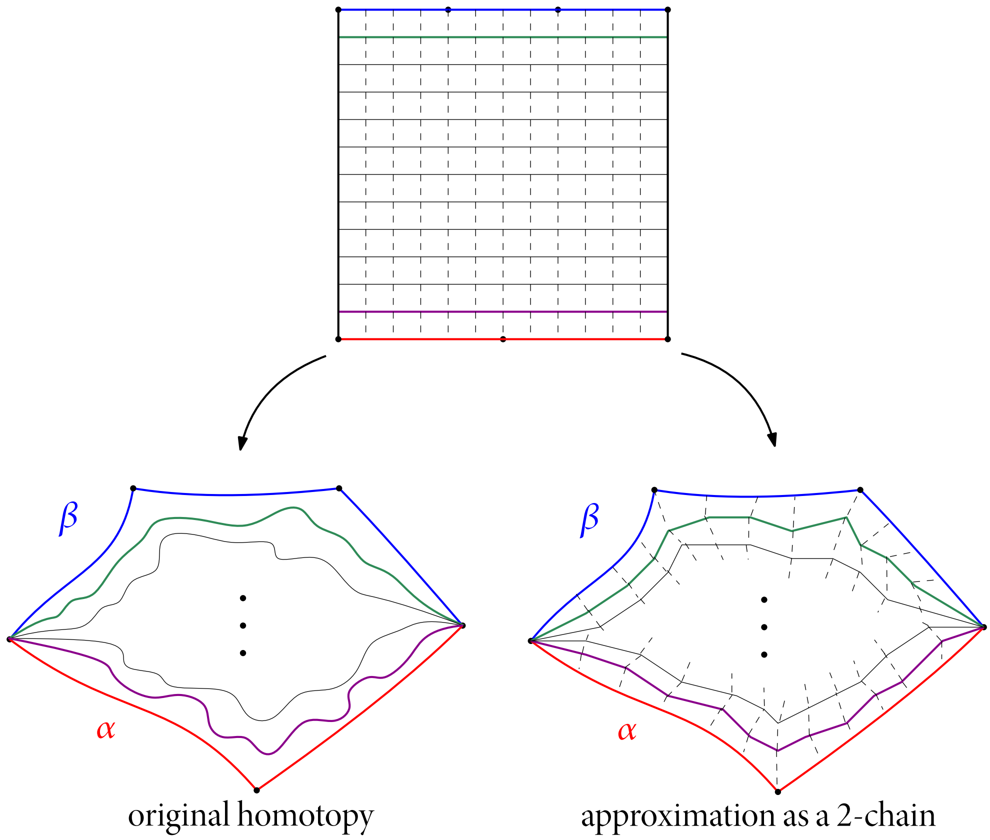

Figure 3.1: Constructing an approximative piecewise C^1 homotopy.

choose an integer N which is a multiple of nm and so large that for $\delta = 1/N$ we have $|H(z) - H(w)| \leq r$ for some fixed $0 < r < \mathrm{dist}(H([0,1]^2), \partial U)$ whenever $|z-w| \leq \sqrt{2}\delta$. (Here $\mathrm{dist}(H([0,1]^2), \partial U) = \infty$ if $\partial U = \emptyset$.) Now, the idea of the rest of the proof is as follows: We divide $[0,1]^2$ into $N\times N$ squares of side length δ, and then define 2-cells corresponding to the small squares by requiring that the boundaries in the interior of $[0,1]^2$ map to line segments whose endpoints are given by the values of the original homotopy H, see Figure 3.1.

More precisely, let us define first 1-cells $I_{j,k}\colon [0,1] \to \mathbb{C}$ $(1 \leq j \leq N, 0 \leq k \leq N)$ by setting

$$I_{j,0}(x) = \alpha((j-1+x)/N), \quad I_{j,N}(x) = \beta((j-1+x)/N)$$

and via horizontal linear interpolation

$$I_{j,k}(x) = (1-x)H((j-1)/N + ik/N) + xH(j/N + ik/N)$$

when $1 \leq k \leq N-1$. Note that since N is a multiple of n and m, each $I_{j,k}$ is C^1.

Using next linear interpolation in the vertical direction we define a 2-chain

$S = \sum_{j,k=1}^{N} S_{j,k}$ with N^2 2-cells $S_{j,k}\colon [0,1]^2 \to \mathbb{C}$, $1 \le j,k \le N$, given by

$$S_{j,k}(x+iy) = (1-y)I_{j,k-1}(x) + yI_{j,k}(x).$$

Notice that every point $z \in S_{j,k}([0,1]^2)$ is a convex combination of at most 4 points of $H([0,1]^2)$ with distance at most r from each other. It follows that $\operatorname{dist}(z, H([0,1]^2)) \le r$ and hence $S_{j,k}([0,1]^2) \subset U$.

It remains to compute the boundary of S. By definition the right boundary of $S_{j,k}$ and the left boundary of $S_{j+1,k}$ cancel each other for $1 \le j \le N-1$ and $1 \le k \le N$, and similarly the top boundary of $S_{j,k}$ and the bottom boundary of $S_{j,k+1}$ cancel for $1 \le j \le N$ and $1 \le k \le N-1$. Moreover, since $H(0,y) = \alpha(0)$ and $H(1,y) = \alpha(1)$ are constant in y, also the left boundary of $S_{0,k}$ and the right boundary of $S_{n,k}$ are constant curves for $1 \le k \le n$ and thus equivalent to 0 as 1-chains. Hence, the boundary of S is equivalent to $\sum_{j=1}^{N}(I_{j,0} - I_{j,n})$ which in turn is equivalent to $\alpha - \beta$. □

As an application we may prove the following.

Theorem 3.18. *Suppose that α and β are contours with common endpoints and homotopic in U. Then for any differentiable 1-form ω in U with $d\omega = 0$ we have*

$$\int_\alpha \omega = \int_\beta \omega.$$

Proof. By Theorem 3.17 there exists a 2-chain S in U with boundary equivalent to $\alpha - \beta$. By Stokes' theorem (Theorem 2.22) we have

$$0 = \int_S d\omega = \int_{\partial S} \omega = \int_\alpha \omega - \int_\beta \omega. \qquad \square$$

As an important corollary we have the following.

Theorem 3.19. *Let α and β be two contours with common endpoints and homotopic in U. Then for any holomorphic function $f\colon U \to \mathbb{C}$ we have*

$$\int_\alpha f(z)\,dz = \int_\beta f(z)\,dz.$$

Proof. Since f is holomorphic, it is real-differentiable and $\partial_{\overline{z}} f(z) = 0$ for $z \in U$. Thus, we have $d(f\,dz) = \partial_{\overline{z}} f\, d\overline{z} \wedge dz = 0$ and the result follows from Theorem 3.18. □

It is useful to also consider homotopies of closed contours where no points are fixed, but the curves stay closed during the homotopy.

Definition 3.20. Let α and β be two closed contours in U. We say that α and β are **loop-homotopic** in U if there exists a continuous function $H\colon [0,1]^2 \to U$ such that

$$\begin{aligned} H(\cdot,0) &= \alpha(\cdot) \\ H(\cdot,1) &= \beta(\cdot) \\ H(0,\cdot) &= H(1,\cdot) \end{aligned}$$ ◆

Theorems 3.16,3.17,3.18 and 3.19 also have analogous versions for loop-homotopies.

Theorem 3.21. *Let α and β be two closed contours in a convex open set U. Then α and β are loop-homotopic.*

Proof. Linear interpolation as in the proof of Theorem 3.16. □

Theorem 3.22. *Let α and β be two closed contours in U that are loop-homotopic to each other. Then there exists a 2-chain in U whose boundary is equivalent to the 1-chain $\alpha - \beta$.*

Proof. The same construction as in Theorem 3.17 works. This time the 1-cells $S_{n,k}|_{[1,1+i]}$ and $S_{0,k}|_{[0,i]}$ are not necessarily constant curves, but they are equal because both are linear interpolations between the same endpoints. We thus have $S_{n,k}|_{[1,1+i]} - S_{0,k}|_{[0,i]} = 0$ as a 1-chain, leaving again only the 1-cells $I_{j,0}$ and $I_{j,n}$ $(1 \le j \le n)$ to contribute to the boundary of the constructed 2-chain. □

Theorem 3.23. *Let α and β be closed contours loop-homotopic to each other in U. Then for any differentiable 1-form ω with $d\omega = 0$ we have*

$$\int_\alpha \omega = \int_\beta \omega.$$

Proof. Analogous to the proof of Theorem 3.18. □

Theorem 3.24. *Let α and β be closed contours loop-homotopic to each other in U. Then for any holomorphic $f\colon U \to \mathbb{C}$ we have*

$$\int_\alpha f\,dz = \int_\beta f\,dz.$$

Proof. Analogous to the proof of Theorem 3.19. □

Definition 3.25. An open set $U \subset \mathbb{C}$ is called **simply connected** if all closed contours in U are loop-homotopic to each other in U. ◆

Intuitively, a set is simply connected if it is connected and does not contain holes. In particular, convex sets are simply connected by Theorem 3.21.

Example 3.26. Consider the contour γ_n given by $\gamma_n(t) = e^{2\pi i n t}$ for some $n \in \mathbb{Z}$, which goes around the unit circle n times. Then

$$\int_{\gamma_n} \frac{1}{z}\, dz = 2\pi i n,$$

which means that γ_n and γ_m are not loop-homotopic in $\mathbb{C} \setminus \{0\}$ unless $n = m$. In particular, $\mathbb{C} \setminus \{0\}$ is not simply connected. ◆

Theorem 3.27 (Cauchy's integral theorem)**.** *Let $f : U \to \mathbb{C}$ be holomorphic and suppose that U is simply connected. Then $\int_\gamma f\, dz = 0$ for every closed contour γ in U.*

Proof. Let $z_0 \in U$ be a fixed point. If $\gamma(t) = z_0$ is the constant loop, then clearly $\int_\gamma f\, dz = 0$. The result then follows from Theorem 3.24. □

3.3 *Cauchy's integral formula*

We will next prove Cauchy's integral formula, which can be used to recover the value of a holomorphic function at a given point via a suitable contour integral.

Theorem 3.28 (Cauchy's integral formula)**.** *Suppose that f is holomorphic in U and $\overline{B}(z_0, r) \subset U$. Then for any $w \in B(z_0, r)$ we have*

$$f(w) = \frac{1}{2\pi i} \int_{\partial B(z_0, r)} \frac{f(z)}{z - w}\, dz.$$

Before going to the proof, let us introduce the following notation

$$\int_\gamma f(z)|dz| := \int_0^1 f(\gamma(t))|\gamma'(t)|\, dt$$

where $\gamma\colon [0,1] \to \mathbb{C}$ is a curve and f is a continuous function defined on $\gamma([0,1])$. Notice that we have by the triangle inequality[2]

$$\left|\int_\gamma f(z)\, dz\right| = \left|\int_0^1 f(\gamma(t))\gamma'(t)\, dt\right| \le \int_0^1 |f(\gamma(t))||\gamma'(t)|\, dt = \int_\gamma |f(z)||dz|$$

and that $\int_\gamma |dz|$ is (by definition) the length of the curve γ. If $\gamma = \gamma_1 + \cdots + \gamma_n$

[2] One can use Riemann sums and the triangle inequality for sums to deduce that $|\int_0^1 g(t)\, dt| = \lim_{N\to\infty} |\sum_{n=1}^N g(n/N)/N| \le \lim_{N\to\infty} \sum_{n=1}^N |g(n/N)|/N = \int_0^1 |g(t)|\, dt$ for any continuous $g\colon [0,1] \to \mathbb{C}$.

is a 1-chain consisting of multiple C^1-arcs, we set similarly

$$\int_\gamma f(z)|dz| = \sum_{k=1}^n \int_{\gamma_k} f(z)|dz|$$

and again have

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)||dz|.$$

Proof of Theorem 3.28. Let $\varepsilon \in (0, r - |w - z_0|)$. Notice that $\partial B(z_0, r)$ is loop-homotopic in $U \setminus \{w\}$ to $\partial B(w, \varepsilon)$, for instance via the homotopy

$$H(t,s) = (1-s)z_0 + sw + ((1-s)r + s\varepsilon)e^{2\pi i t}.$$

Indeed, for any $t, s \in [0,1]$ the triangle inequality and the upper bound on ε gives us

$$|H(t,s) - z_0| = |s(w - z_0) + ((1-s)r + s\varepsilon)e^{2\pi i t}| \le s|w - z_0| + (1-s)r + s\varepsilon < r,$$

so that $H([0,1]^2) \subset \overline{B}(z_0, r) \subset U$. Moreover, by the reverse triangle inequality $|a + b| \ge |b| - |a|$ we have

$$\begin{aligned}|H(t,s) - w| &= |(1-s)(z_0 - w) + ((1-s)r + s\varepsilon)e^{2\pi i t}| \\ &\ge |((1-s)r + s\varepsilon)e^{2\pi i t}| - (1-s)|z_0 - w| \\ &= (1-s)(r - |z_0 - w|) + s\varepsilon \ge (1-s)\varepsilon + s\varepsilon = \varepsilon > 0,\end{aligned}$$

so that $w \notin H([0,1]^2)$. By Theorem 3.24 and Lemma 3.14 we then have that

$$\left|\frac{1}{2\pi i}\int_{\partial B(z_0,r)} \frac{f(z)}{z-w}\,dz - f(w)\right| = \left|\frac{1}{2\pi i}\int_{\partial B(w,\varepsilon)} \frac{f(z) - f(w)}{z - w}\,dz\right|.$$

Note that since f is complex-differentiable at w, we have

$$\left|\frac{f(z) - f(w)}{z - w}\right| \le |f'(w)| + 1 =: C$$

for $|z - w|$ small enough. Thus, for small enough ε we have

$$\left|\frac{1}{2\pi i}\int_{\partial B(w,\varepsilon)} \frac{f(z) - f(w)}{z - w}\,dz\right| \le \frac{1}{2\pi}\int_{\partial B(w,\varepsilon)} C\,|dz| = C\varepsilon$$

and the claim follows by letting $\varepsilon \to 0$. □

We will next use Cauchy's integral formula to prove the very important result that derivatives of holomorphic functions are themselves holomorphic and have an analogous integral representation.

Theorem 3.29 (Generalized Cauchy's formula)**.** *Suppose that f is holomorphic in U. Then f is n-times $\mathbb{C}$-differentiable on U for any $n \geq 0$ and if $z_0 \in U$ and $r > 0$ are such that $\overline{B}(z_0, r) \subset U$, we have*

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(z)}{(z-w)^{n+1}}\, dz$$

for any $w \in B(z_0, r)$.

Proof. We will show by induction that for any $w \in B(z_0, r)$ and $n \geq 0$ the derivative $f^{(n)}(w)$ exists and equals

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(z)}{(z-w)^{n+1}}\, dz.$$

We know from Theorem 3.28 that the claim holds for $n = 0$, so the initial step is clear. Suppose thus that it holds for some $n \geq 0$. We can write the difference quotient of $f^{(n)}$ as

$$\frac{f^{(n)}(w+h) - f^{(n)}(w)}{h} = \frac{n!}{2\pi i} \int_{\partial B(z_0,r)} \frac{a^{n+1} - b^{n+1}}{h} f(z)\, dz,$$

where $a = 1/(z - w - h)$ and $b = 1/(z - w)$. Recall that $a^{n+1} - b^{n+1} = (a - b)(a^n + a^{n-1}b + \cdots + b^{n-1}a + b^n)$ and note that $a - b = hab$, so that we have

$$\frac{a^{n+1} - b^{n+1}}{h} = ab(a^n + a^{n-1}b + \cdots + b^{n-1}a + b^n).$$

Note that this tends to $(n+1)(z-w)^{-(n+2)}$ as $h \to 0$, and moreover is uniformly bounded for $z \in \partial B(z_0, r)$, since for small enough h we have $|z - w - h| > |z - w|/2 > (r - |w - z_0|)/4$. It follows that the integral converges as $h \to 0$, $f^{(n)}$ is complex-differentiable at w, and

$$f^{(n+1)}(w) = \frac{(n+1)!}{2\pi i} \int_{\partial B(z_0,r)} \frac{f(z)}{(z-w)^{n+2}}\, dz$$

as wanted. □

Let us close this section with an example computation of a concrete contour integral using Cauchy's integral formula.

Example 3.30. Let us evaluate

$$\int_{\partial B(0,2)} \frac{(z^4+1)\sin(z)}{z^3-iz^2}\,dz.$$

Note first that by factoring the denominator we have

$$\frac{(z^4+1)\sin(z)}{z^3-iz^2} = \frac{(z^4+1)\sin(z)}{z^2(z-i)},$$

which is analytic in $\mathbb{C}\setminus\{0,1\}$. We may next use a partial fraction decomposition (see below) to write

$$\frac{z^4+1}{z^2(z-i)} = z + i + \frac{i}{z^2} + \frac{1}{z} - \frac{2}{z-i}.$$

Note that since $(z+i)\sin(z)$ is holomorphic in the whole plane, we have

$$\int_{\partial B(0,2)} (z+i)\sin(z)\,dz = 0$$

by Cauchy's integral theorem. For the other three terms (the generalized) Cauchy's integral formula gives us

$$\begin{aligned}
&\int_{\partial B(0,2)} \frac{(z^4+1)\sin(z)}{z^3-iz^2}\,dz \\
&= \int_{\partial B(0,2)} \frac{i\sin(z)}{z^2}\,dz + \int_{\partial B(0,2)} \frac{\sin(z)}{z}\,dz - \int_{\partial B(0,2)} \frac{2\sin(z)}{z-i}\,dz \\
&= 2\pi i(i\sin'(0) + \sin(0) - 2\sin(i)) = -2\pi - 4\pi i\sin(i) \\
&= -2\pi - 4\pi i\frac{e^{i\cdot i} - e^{-i\cdot i}}{2i} = 2\pi(e - e^{-1} - 1).
\end{aligned}$$

◆

In the previous example we used a partial fraction decomposition. In general, if we have a rational function of the form $\frac{p(z)}{(z-a_1)^{k_1}\ldots(z-a_n)^{k_n}}$ where p is a polynomial, $a_1,\ldots,a_n$ are distinct complex numbers and $k_1,\ldots,k_n \geq 1$ are integers, we can write it in the form

$$q(z) + \sum_{\ell=1}^{n}\sum_{j=1}^{k_\ell} \frac{c_{\ell,j}}{(z-a_\ell)^j}$$

where $q(z) = q_d z^d + \cdots + q_1 z + q_0$ is a polynomial of degree $d = \deg(p) - k_1 - \cdots - k_n$ (we set $q = 0$ if $d < 0$) and $c_{\ell,j}$ are complex numbers. To find q and $c_{\ell,j}$, one can multiply the above expression by the denominator $(z-a_1)^{k_1}\ldots(z-a_n)^{k_n}$

of the original function to get a polynomial whose coefficients have to equal those of p. This gives us a linear system for the unknowns q_k and $c_{\ell,j}$.

For instance, consider the function $\frac{z^4+1}{z^2(z-i)}$ we had above. Since the numerator has degree 4 which is one more than the degree of the denominator, q should have degree 1, and we look for constants a, b, c, d, e such that

$$\frac{z^4+1}{z^2(z-i)} = az + b + \frac{c}{z^2} + \frac{d}{z} + \frac{e}{z-i}.$$

Multiplying by $z^2(z-i)$ we get

$$\begin{aligned} z^4 + 1 &= az^3(z-i) + bz^2(z-i) + c(z-i) + dz(z-i) + ez^2 \\ &= az^4 + (b-ia)z^3 + (d+e-ib)z^2 + (c-id)z - ic, \end{aligned}$$

Equating the coefficients we get the equations $a = 1, b - ia = 0, d + e - ib = 0$, $c - id = 0$ and $-ic = 1$, whose solution is $a = 1, b = i, c = i, d = 1, e = -2$, giving us

$$\frac{z^4+1}{z^2(z-i)} = z + i + \frac{i}{z^2} + \frac{1}{z} - \frac{2}{z-i}.$$

3.4 *Morera's theorem*

Morera's theorem provides a converse to Cauchy's integral theorem: If integrating f over closed contours always gives 0 as the result, then f is holomorphic. In fact, it is enough to check this for triangular contours.

Theorem 3.31 (Morera's theorem)**.** *Let f be a continuous function on U and suppose that*

$$\int_{\partial T} f(z)\, dz = 0$$

for all closed triangles T in U. Then f is holomorphic in U.

Proof. It is enough to prove the theorem locally, i.e. inside a ball $B(z_0, r)$ for some $z_0 \in U$ and $r > 0$. Let us define $F\colon B(z_0, r) \to \mathbb{C}$ by setting

$$F(z) = \int_{z_0}^{z} f(w)\, dw,$$

where the integral is along a straight segment from z_0 to z. If we can show that F is holomorphic in $B(z_0, r)$ and $F'(z) = f(z)$, we will be done since derivatives of holomorphic functions are holomorphic by Theorem 3.29. Suppose that $z \in B(z_0, r)$ and that $h \in \mathbb{C}$ has small enough modulus so that $z + h \in B(z_0, r)$ as well. Then the triangle with vertices z_0, z and $z + h$ lies inside $B(z_0, r)$ by convexity, and we have

$$\frac{F(z+h)-F(z)}{h} = \frac{\int_{z_0}^{z+h} f(w)\,dw - \int_{z_0}^{z} f(w)\,dw}{h}.$$

By assumption, we have

$$\int_{z_0}^{z+h} f(w)\,dw = \int_{z_0}^{z} f(w)\,dw + \int_{z}^{z+h} f(w)\,dw,$$

so that

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| \le \frac{1}{h}\left|\int_{z}^{z+h} (f(w)-f(z))\,dw\right|,$$

which tends to 0 as $h \to 0$ by the continuity of f. □

As a corollary we get the following result on uniform limits of holomorphic functions.

Theorem 3.32. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of holomorphic functions $f_n : U \to \mathbb{C}$ converging uniformly on compact subsets of U to a function $f : U \to \mathbb{C}$. Then f is holomorphic in U.*

Proof. It is enough to show that f is holomorphic in any ball $B(z_0, r) \subset U$. Let T be a closed triangle in $B(z_0, r)$. Since $f_n \to f$ uniformly on compact subsets, f is continuous, and we may exchange taking limits and integration to get

$$\int_{\partial T} f(z)\,dz = \int_{\partial T} \lim_{n\to\infty} f_n(z)\,dz = \lim_{n\to\infty} \int_{\partial T} f_n(z)\,dz = 0$$

by Cauchy's integral theorem (note that $B(z_0, r)$ is simply connected), making Morera's theorem applicable. □

Power series

4.1 *Analytic functions*

Definition 4.1. A function $f : U \to \mathbb{C}$ is **analytic** in U, if for every $z_0 \in U$ there exists $r > 0$ and coefficients $(a_n)_{n=0}^{\infty}$ in $\mathbb{C}$ such that f can be written as a converging power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for $z \in B(z_0, r) \subset U$. ◆

The main result of this section is that analytic functions are holomorphic and vice versa. Indeed, often the two terms are used synonymously.

Let us begin with the following lemma.

Lemma 4.2. *Suppose that the power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges in the open disc $B(z_0, R)$ for some $R > 0$. Then the convergence is uniform in any smaller disc $B(z_0, r)$ with $r < R$.*

Proof. By translation and scaling it is enough to consider the case $z_0 = 0$ and $R = 1$. We first claim that for any $\varepsilon > 0$ we have $|a_n| \leq (1 + \varepsilon)^n$ for n large enough. Assume that this is not the case. Then there exist infinitely many n such that $|a_n| > (1+\varepsilon)^n$. But then for $z = 1/(1+\varepsilon)$ we have $|a_n z^n| > 1$ infinitely often, and the series $\sum_{n=0}^{\infty} a_n z^n$ cannot converge, giving a contradiction.

Let now $r < 1$ and fix $\varepsilon > 0$ so small that $(1 + \varepsilon)r < 1$. We have

$$\sup_{z \in B(z_0, r)} \left| \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{N} a_n z^n \right| \leq \sum_{n=N+1}^{\infty} |a_n| r^n \leq \sum_{n=N+1}^{\infty} ((1 + \varepsilon) r)^n$$

for N large enough. The right-hand side tends to 0 as $N \to \infty$, showing uniform convergence in $B(z_0, r)$. □

Theorem 4.3. *Let f be an analytic function in U. Then f is holomorphic in U and the power series of f around $z_0 \in U$ is unique and given by*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

Proof. It is enough to show that f is holomorphic in a neighborhood of any point $z_0 \in U$. Since f is analytic, we can find $r > 0$ and coefficients $a_n \in \mathbb{C}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

holds for $z \in B(z_0, r)$. By Lemma 4.2 the series converges uniformly on the ball $B(z_0, r/2)$. Since each partial sum $\sum_{n=0}^{N} a_n (z - z_0)^n$ is a polynomial and hence holomorphic, the holomorphicity of f follows from Theorem 3.32. By Cauchy's integral formula, uniform convergence and Lemma 3.14 we also have

$$\begin{aligned} f^{(k)}(z_0) &= \frac{k!}{2\pi i} \int_{\partial B(z_0, r/2)} \frac{\sum_{n=0}^{\infty} a_n (z - z_0)^n}{(z - z_0)^{k+1}} \, dz \\ &= \sum_{n=0}^{\infty} \frac{k!}{2\pi i} \int_{\partial B(z_0, r/2)} a_n (z - z_0)^{n-k-1} \, dz = k! a_k \end{aligned}$$

showing that we necessarily have $a_k = f^{(k)}(z_0)/k!$. □

Let us next prove the converse.

Theorem 4.4. *Let f be a holomorphic function in some disc $B(z_0, R)$. Then we have*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

for all $z \in B(z_0, R)$.

Proof. Let $z \in B(z_0, R)$ and pick $r \in (|z - z_0|, R)$. Since f is holomorphic, we have by Cauchy's integral formula that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial B(z_0, r)} \frac{f(w)}{w - z} \, dw = \frac{1}{2\pi i} \int_{\partial B(z_0, r)} \frac{f(w)}{w - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{w - z_0}} \, dw \\ &= \frac{1}{2\pi i} \int_{\partial B(z_0, r)} \frac{f(w)}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n \, dw, \end{aligned}$$

where the geometric series converges since $|z - z_0| < |w - z_0| = r$. For fixed z and r the series converges uniformly in w, and we can switch the order of integration and summation to get

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{\partial B(z_0, r)} \frac{f(w)}{(w - z_0)^{n+1}} \, dw \cdot (z - z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

by the generalized Cauchy's integral formula. □

Corollary 4.5. *Let f be a holomorphic function in an open set U. Then f is analytic in U and the radius of convergence of the power series of f around $z_0 \in U$ is at least* $\operatorname{dist}(z_0, \partial U)$.

Remark. Series of the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

where the coefficients are given by nth order derivatives of a holomorphic function divided by $n!$ are also called **Taylor series**. Thus, we have shown that any power series with positive radius of convergence defines a holomorphic function whose Taylor series agrees with the original power series. ◆

Let us also note the following simple version of Taylor's theorem for analytic functions.

Lemma 4.6. *Suppose that f has a converging power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for $z \in B(z_0, R)$. Then for any $k \geq 0$ we have the asymptotic expansion

$$f(z) = \sum_{n=0}^{k} a_n (z - z_0)^n + O((z - z_0)^{k+1})$$

as $z \to z_0$.

Proof. Let

$$g(z) := \sum_{n=k+1}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} a_{n+k+1} (z - z_0)^{n+k+1}$$

so that

$$f(z) = \sum_{n=0}^{k} a_n (z - z_0)^n + g(z).$$

Note the following simple fact: For any coefficients c_n and constant $\lambda \neq 0$ a complex series $\sum_{n=0}^{\infty} c_n$ converges to L if and only if $\sum_{n=0}^{\infty} \lambda c_n$ converges to λL. Thus, applying this with $c_n = a_{n+k+1}$ and $r = (z - z_0)^{k+1}$, we see that for a fixed $z \neq z_0$ the series

$$h(z) := \sum_{n=0}^{\infty} a_{n+k+1} (z - z_0)^n$$

converges if and only if the series $g(z)$ converges, which we know happens in $B(z_0, R)$. Moreover, $h(z)$ clearly converges for $z = z_0$ as well, showing that h is an analytic function in $B(z_0, R)$ and

$$g(z) = (z - z_0)^{k+1} h(z).$$

In particular, $h(z)$ is bounded as $z \to z_0$, implying that $g(z) = O((z - z_0)^{k+1})$ as $z \to z_0$. □

Sometimes we are given a power series in terms of its coefficients, and we would like to know its radius of convergence. The answer is given by the following theorem.

Theorem 4.7. *Let $(a_n)_{n=0}^{\infty}$ be a sequence of complex numbers and $z_0 \in \mathbb{C}$. Then the power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges in $B(z_0, R)$, where*

$$R = \liminf_{n \to \infty} |a_n|^{-1/n}.$$

The series does not converge for any z with $|z - z_0| > R$. For $|z - z_0| = R$ the series may or may not converge.

Proof. We may without loss of generality assume that $z_0 = 0$. Suppose first that $|z| < r$ for some $r < R$. By assumption $|a_n|^{-1/n} > r$ for n large enough, meaning that

$$\sum_{n=0}^{\infty} |a_n||z|^n \leq \sum_{n=0}^{\infty} r^{-n}|z|^n < \infty,$$

so that the series converges absolutely.

Suppose next that $|z| > r > R$. By assumption, we have $|a_n|^{-1/n} < r$ for infinitely many n. This implies that we have $|a_n||z|^n \geq r^{-n} r^n = 1$ for infinitely many n, which means that the series cannot converge at z. □

Let us close this section with an application to real Taylor series.

Example 4.8. Consider the function $f(x) = \frac{1}{1+x^2}$ on the real line. We can write its Taylor series at some point $x_0 > 0$. It turns out that one can express the series as

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \operatorname{Im}((x_0 - i)^{-n-1})(x - x_0)^n,$$

but without even computing the terms, we know that its radius of convergence will be $\sqrt{1 + x_0^2}$, since this is the distance from x_0 to the boundary of the open set $\mathbb{C} \setminus \{\pm i\}$ where f is holomorphic.

It came to appear that, between two truths of the real domain, the easiest and shortest path quite often passes through the complex domain. –Paul Painlevé ◆

4.2 *Laurent series*

Laurent series are a generalization of power series where we allow negative powers of $z - z_0$ in addition to the non-negative ones.

Definition 4.9. A **Laurent series** is a series of the form

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where $a_n, z_0 \in \mathbb{C}$ and z is a variable. ◆

A Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ can be split into a sum $f(z) = g(z) + h(z)$, where

$$g(z) = \sum_{n=-\infty}^{-1} a_n (z - z_0)^n \quad \text{and} \quad h(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

The series $g(z)$ containing the terms of negative order is sometimes called the **principal part** of $f(z)$. The power series $h(z)$ converges for $|z| < R_h := \liminf_{n\to\infty} |a_n|^{-1/n}$, while $g(z)$ can be viewed as a power series $\sum_{n=1}^{\infty} a_{-n} w^n$ in $w = 1/(z - z_0)$ that converges when

$$|w| < \liminf_{n\to\infty} |a_{-n}|^{-1/n}.$$

This can be equivalently written as

$$|z - z_0| > R_g := \limsup_{n\to\infty} |a_{-n}|^{1/n}.$$

Hence, if $R_g < R_h$, we see that the Laurent series $f(z)$ converges in the annulus $A(z_0, R_g, R_h)$ to a holomorphic function. Conversely, we have the following.

Theorem 4.10. *Let $z_0 \in \mathbb{C}$ and $0 \le r < R \le \infty$. Suppose that $f\colon A(z_0, r, R) \to \mathbb{C}$ is holomorphic.*[1] *Then f has a converging Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

for $z \in A(z_0, r, R)$, where the coefficients a_n are given by

$$a_n = \frac{1}{2\pi i} \int_{\partial B(z_0,\rho)} \frac{f(z)}{(z - z_0)^{n+1}}\, dz,$$

independent of the radius $\rho \in (r, R)$.

[1] Note that we have $A(z_0, 0, R) = B(z_0, R) \setminus \{z_0\}$ if $r = 0$.

Before proving the theorem let us show the following version of Cauchy's integral formula for annuli.

Lemma 4.11. *Let $f : A(z_0, r, R) \to \mathbb{C}$ be holomorphic and let a and b be such that $r < a < b < R$. Then for any $z \in A(z_0, a, b)$ we have*

$$f(z) = \frac{1}{2\pi i} \int_{\partial A(z_0,a,b)} \frac{f(w)}{z - w}\, dw.$$

Proof. Let $g(w) = \frac{f(w)-f(z)}{w-z}$ for $w \in A(z_0, r, R) \setminus \{z\}$ and $g(z) = f'(z)$. Then g is holomorphic in $A(z_0, r, R)$ since it is clearly holomorphic in $A(z_0, r, R) \setminus \{z\}$ and by analyticity of f at z we can write for $w \neq z$,

$$\begin{aligned} g(w) &= \frac{f(z) + f'(z)(w-z) + \frac{f''(z)}{2}(w-z)^2 + O((w-z)^3) - f(z)}{w-z} \\ &= f'(z) + \frac{f''(z)}{2}(w-z) + O((w-z)^2), \end{aligned}$$

so that g is $\mathbb{C}$-differentiable at z. In particular, by Stokes' theorem we have $\int_{\partial A(z_0,a,b)} g(w)\, dw = 0$ and thus,

$$\int_{\partial A(z_0,a,b)} \frac{f(w)}{w-z}\, dw = \int_{\partial A(z_0,a,b)} \frac{f(z)}{w-z}\, dw = f(z) \int_{\partial A(z_0,a,b)} \frac{1}{w-z}\, dw.$$

Finally, we note that

$$\int_{\partial A(z_0,a,b)} \frac{1}{w-z}\, dw = \int_{\partial B(z_0,b)} \frac{1}{w-z}\, dw - \int_{\partial B(z_0,a)} \frac{1}{w-z}\, dw = 2\pi i$$

since $B(z_0, b)$ contains z but $B(z_0, a)$ does not. □

Proof of Theorem 4.10. We may assume without loss of generality that $z_0 = 0$. Using Lemma 4.11 we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial A(0,a,b)} \frac{f(w)}{w-z}\, dw = \frac{1}{2\pi i} \int_{\partial B(0,b)} \frac{f(w)}{w-z}\, dw - \frac{1}{2\pi i} \int_{\partial B(0,a)} \frac{f(w)}{w-z}\, dw.$$

Note that when $w \in \partial B(0,b)$ we have $|z/w| < 1$, so we may write

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial B(0,b)} \frac{f(w)}{w-z}\,dw &= \frac{1}{2\pi i}\int_{\partial B(0,b)} \frac{f(w)}{w}\cdot\frac{1}{1-\frac{z}{w}}\,dw \\
&= \frac{1}{2\pi i}\int_{\partial B(0,b)} \frac{f(w)}{w}\sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^n dw. \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{\partial B(0,b)} \frac{f(w)}{w^{n+1}}\,dw\, z^n \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{\partial B(0,\rho)} \frac{f(w)}{w^{n+1}}\,dw\, z^n.
\end{aligned}$$

Here we used uniform convergence to exchange summation and integration and homotopy invariance to move the contour to the circle of radius $\rho \in (r,R)$. This gives us the non-negative terms in the Laurent series. Similarly, when $w \in \partial B(0,a)$ we have $|w/z| < 1$, and we may write

$$\begin{aligned}
-\frac{1}{2\pi i}\int_{\partial B(0,a)} \frac{f(w)}{w-z}\,dw &= \frac{1}{2\pi i}\int_{\partial B(0,a)} \frac{f(w)}{z}\cdot\frac{1}{1-\frac{w}{z}}\,dw \\
&= \frac{1}{2\pi i}\int_{\partial B(0,a)} \frac{f(w)}{z}\sum_{n=0}^{\infty}\left(\frac{w}{z}\right)^n dw. \\
&= \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{\partial B(0,a)} f(w)w^n\,dw\, z^{-n-1} \\
&= \sum_{n=-\infty}^{-1}\frac{1}{2\pi i}\int_{\partial B(0,\rho)} \frac{f(w)}{w^{n+1}}\,dw\, z^n,
\end{aligned}$$

which gives us the negative terms in the Laurent series. □

Functions f that are analytic in a punctured disc $B(z_0,r)\setminus\{z_0\}$ can be classified based on their Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ around z_0:

- If
$$f(z) = \sum_{n=-m}^{\infty} a_n(z-z_0)^n$$
with $m \in \mathbb{Z}$ and $a_{-m} \neq 0$, then f is said to have a **pole** of order m at z_0.
- In the above case we also say that f has a **zero** of order $-m$ at z_0.
- If $m \leq 0$, then f is said to have a **removable singularity** at z_0, and f can be extended to an analytic function
$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

in $B(z_0, r)$.

- If $a_{-n} \neq 0$ for infinitely many $n \geq 1$, then f is said to have an **essential singularity** at z_0.

Lemma 4.12. *Suppose that f is holomorphic in $B(z_0, r) \setminus \{z_0\}$. Then f has a pole of order $n \in \mathbb{Z}$ at z_0 if and only if $(z - z_0)^n f(z)$ is bounded near z_0 but $(z - z_0)^{n-1} f(z)$ is unbounded. In this case $(z - z_0)^n f(z)$ will extend analytically to $B(z_0, r)$.*

Proof. Clearly if f has a pole of order n at z_0, then $h(z) = (z - z_0)^n f(z)$ will be analytic near z_0 and hence bounded, while $(z - z_0)^{n-1} f(z)$ will not be bounded.

Conversely, suppose that $h(z)$ is bounded in $B(z_0, r) \setminus \{z_0\}$ and $(z - z_0)^{n-1} f(z)$ is unbounded. Then if a_n are the coefficients of the Laurent series of $h(z)$, we have for $n \geq 1$ that

$$|a_{-n}| \leq \frac{1}{2\pi} \int_{\partial B(z_0, \rho)} |h(z)||z - z_0|^{n-1} \, |dz|$$

which tends to 0 as $\rho \to 0$. Hence, the principal part of $h(z)$ vanishes and h can be extended analytically to $B(z_0, r)$. Thus,

$$f(z) = \sum_{k=-n}^{\infty} a_{k+n}(z - z_0)^k$$

and f has a pole of order n. □

Example 4.13. The function $f(z) = \frac{e^{1/z}}{(z-2)(z+i)^3}$ has an essential singularity at 0, a first order pole at 2 and a third order pole at $-i$. To see this, we first note that f is unbounded near all three points, so none of the singularities are removable. It is also easy to see by considering real $z > 0$ that $z^n f(z)$ is unbounded as $z \to 0$ for any $n \geq 1$, so the singularity at 0 is essential. Similarly, $(z - 2) f(z)$ and $(z + i)^3 f(z)$ are bounded as $z \to 2$ or $z \to -i$ (but $(z + i)^2 f(z)$ is not), so f has first and third order poles at these points. ◆

It is important to note that the Laurent series of a function really depends on the annulus, not just the center point z_0. For instance, the function $f(z) = \frac{1}{z-1}$ has two different Laurent series around 0. The first one is the geometric series

$$-\sum_{n=0}^{\infty} z^n$$

converging in $B(0, 1)$, while the second one converges in $A(0, 1, \infty)$ and can be found by writing

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=-\infty}^{-1} z^n.$$

4.3 *Computing series*

In this section we will illustrate some computational methods for finding Taylor and Laurent series. Let us first note that Laurent series behave well under differentiation.

Theorem 4.14. *Suppose that f has a converging Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

in some annulus $A(z_0, r, R)$. Then the Laurent series of f' in $A(z_0, r, R)$ is given by

$$\sum_{n=-\infty}^{\infty} (n+1)a_{n+1}(z-z_0)^n.$$

Proof. Let $\rho \in (r, R)$ and let b_n be the coefficients of the Laurent series of f' in $A(z_0, r, R)$, so that

$$b_n = \frac{1}{2\pi i}\int_{\partial B(z_0,\rho)} \frac{f'(z)}{(z-z_0)^{n+1}}\, dz.$$

We will do integration by parts by noting that

$$\frac{f'(z)}{(z-z_0)^{n+1}}\, dz = d\left(\frac{f(z)}{(z-z_0)^{n+1}}\right) + \frac{(n+1)f(z)}{(z-z_0)^{n+2}}\, dz$$

so that by Stokes' theorem we have

$$b_n = \frac{1}{2\pi i}\int_{\partial B(z_0,\rho)} \frac{(n+1)f(z)}{(z-z_0)^{n+2}}\, dz = (n+1)a_{n+1}.$$ □

Let us next discuss some common methods for finding Taylor or Laurent series.

- In the case of Taylor series, computing derivatives.
  - Example: Taylor series $\sum_{n=0}^{\infty} a_n(z-a)^n$ of $\sin(z)$ around $z = a$ has coefficients $a_n = \sin(a)/n!$ if $n \equiv 0 \pmod 4$, $a_n = \cos(a)/n!$ if $n \equiv 1 \pmod 4$, $a_n = -\sin(a)/n!$ if $n \equiv 2 \pmod 4$ and $a_n = -\cos(a)/n!$ if $n \equiv 3 \pmod 4$.

– Example: Let $f(z) = \mathrm{Log}(1-z)$. Then $f'(z) = -1/(1-z)$, $f''(z) = -1/(1-z)^2$, $f^{(3)}(z) = -2/(1-z)^3$, $f^{(4)}(z) = -3!/(1-z)^4$ and in general $f^{(n)}(z) = -(n-1)!/(1-z)^n$. Hence,

$$\mathrm{Log}(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}.$$

- Using known series as building blocks.
  – Example: The Taylor series of e^{z^2} at $z = 0$ is given by $\sum_{n=0}^{\infty} \frac{z^{2n}}{n!}$.
  – Example: Consider $f(z) = \frac{1}{(1-z)^2}$. Then $f(z) = g'(z)$, where $g(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ and hence $f(z) = \sum_{n=0}^{\infty}(n+1)z^n$.
- Especially when only finitely many leading order terms are needed, multiplication or division of known expansions while keeping track of the remainder (using O-notation or otherwise) can be powerful.
  – Example: Consider $f(z) = 1/\sin(z)$ around $z = 0$. We know that $\sin(z) = z - \frac{z^3}{3!} + O(z^5)$. We can do the division by extracting the leading order asymptotics of f one-by-one to get

$$\begin{aligned}\frac{1}{\sin(z)} &= \frac{1}{z - \frac{z^3}{3!} + O(z^5)} = \frac{1}{z} + \frac{\frac{z^2}{3!} + O(z^4)}{z - \frac{z^3}{3!} + O(z^5)} \\ &= \frac{1}{z} + \frac{\frac{z}{3!} + O(z^3)}{1 - \frac{z^2}{3!} + O(z^4)} = \frac{1}{z} + \frac{z}{3!} + \frac{\frac{z^3}{3!^2} + O(z^3)}{1 - \frac{z^2}{3!} + O(z^5)} \\ &= \frac{1}{z} + \frac{z}{6} + O(z^3).\end{aligned}$$

Residue calculus

5.1 *Winding numbers and boundary chains*

So far we have seen versions of Cauchy's integral formula for the disc and the annulus. The residue theorem we are going to prove will be a generalization of them, where instead of $f(z)/(z - z_0)$ with first order pole at z_0 we are allowed to have functions with multiple singularity points, and where we integrate over the boundaries of more general sets (for instance regions with multiple holes).

We will from now on assume that the coefficients of our n-chains are integers.

Definition 5.1. A 1-chain γ in U is called **closed** if $\partial\gamma = 0$, and a **boundary** in U if there exists a 2-chain S in U such that ∂S is equivalent with γ. ◆

It is easy to see that if γ is a boundary then it is automatically closed since for a single 2-cell its boundary is a closed contour. We have also seen that if U is simply connected then every closed contour is a boundary. It is in fact quite easy to show that a closed 1-chain (with integer coefficients) is always a sum of closed contours, so the same holds for closed 1-chains as well.

Our goal in this section is to show the topological result that boundaries in U are exactly the 1-chains whose total rotation, or *winding*, around every $z \in \mathbb{C} \setminus U$ is 0. For instance, an annulus $A(0, 1, 2)$ has two boundary components, $-\partial B(0, 1)$ and $\partial B(1, 2)$, where the minus sign indicates that the inner boundary is oriented clockwise. For points $z \in B(0, 1)$ the boundary $-\partial B(0, 1)$ makes one turn clockwise around z while $\partial B(0, 2)$ makes one turn counter-clockwise, so the total winding is 0. When z lies inside the annulus, the inner boundary has winding 0 and the outer boundary has winding 2π, so the total winding is 2π. Finally, for points z with $|z| > 2$ both the inner and the outer boundary has winding 0. Thus, $\partial A(0, 1, 2)$ is a boundary in $A(0, 1/2, 5/2)$, for instance.

Let us next define winding rigorously. Since it does not make sense to measure the winding at a point that lies on the 1-chain, it is useful to have a notation for the total image of the cells of a 1-chain.

Definition 5.2. Suppose that $\gamma = c_1\gamma_1 + \cdots + c_n\gamma_n$ is a 1-chain with coefficients $c_1, \ldots, c_n \in \mathbb{Z} \setminus \{0\}$ and pairwise distinct 1-cells $\gamma_1, \ldots, \gamma_n$.[1] The **support** of γ is defined as $\operatorname{supp}(\gamma) \coloneqq \bigcup_{k=1}^{n} \gamma_k([0, 1])$. ◆

We can now define the winding of a 1-chain γ around a given point. Let

[1] Note that every 1-chain can be uniquely written in this form.

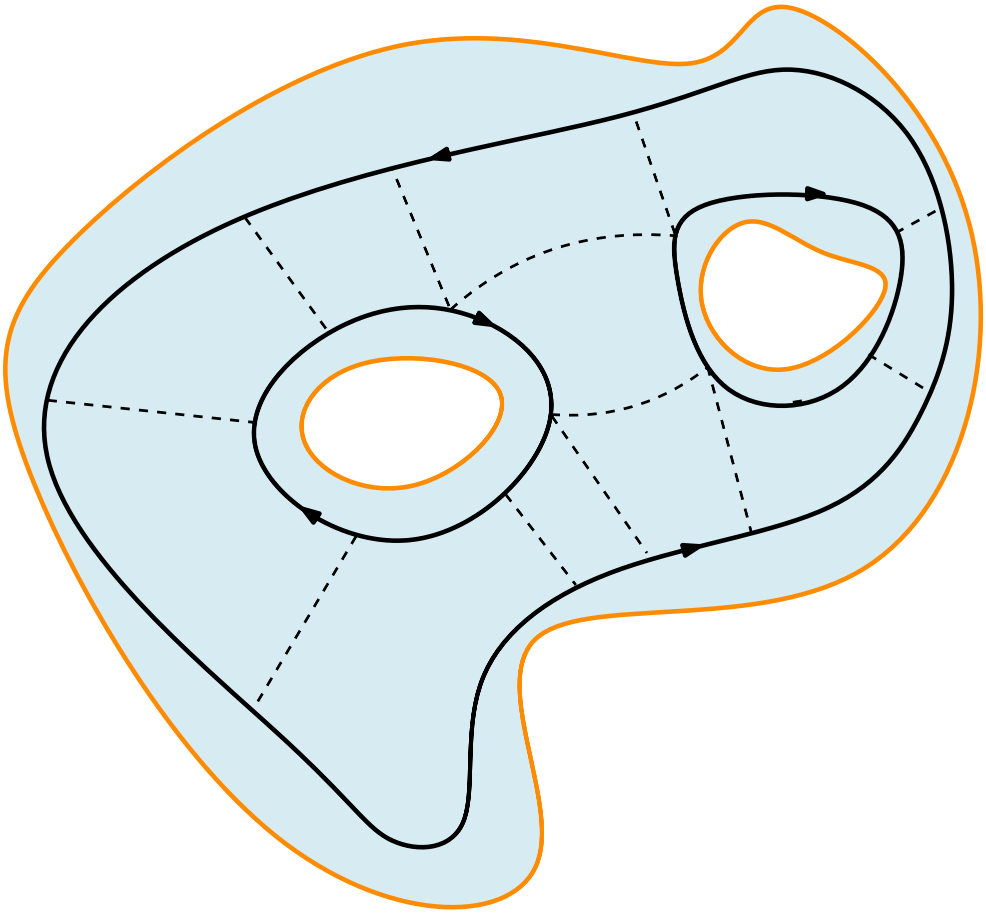

Figure 5.1: A 1-chain (solid black) in an open set U (light blue) that is the boundary of a 2-chain (dashed black) in U.

(r_{z_0}, θ_{z_0}) denote the polar coordinates centered at the point $z_0 \notin \operatorname{supp}(\gamma)$, i.e. $r_{z_0}(z) = |z - z_0|$ and $\theta_{z_0}(z) = \operatorname{Arg}(z - z_0) = \operatorname{Im}(\operatorname{Log}(z - z_0))$, where Arg is the principal branch of the argument. Note that these coordinates are C^1 in the set $z - z_0 \in \mathbb{C} \setminus (-\infty, 0]$, in which case[2]

$$d\theta_{z_0} = \operatorname{Im}(d \operatorname{Log}(z - z_0)) = \operatorname{Im}\left(\frac{dz}{z - z_0}\right).$$

The right-hand side however is well-defined and C^1 also when $z - z_0 \in (-\infty, 0)$, so we can use this formula to define $d\theta_{z_0}$ as a 1-form in $\mathbb{C} \setminus \{z_0\}$.[3]

Definition 5.3. Let γ be a 1-chain and let $z_0 \in \mathbb{C} \setminus \operatorname{supp}(\gamma)$. The **winding** $W(\gamma, z_0)$ of γ around z_0 is given by

$$W(\gamma, z_0) = \int_\gamma d\theta_{z_0} = \operatorname{Im}\left(\int_\gamma \frac{dz}{z - z_0}\right),$$

where $d\theta_{z_0} := \operatorname{Im}\left(\frac{dz}{z-z_0}\right)$. We also define the **index** $\operatorname{Ind}(\gamma, z_0) := W(\gamma, z_0)/(2\pi)$ to be the amount of winding measured in whole turns. ◆

Theorem 5.4. *Let γ be a closed 1-chain with integer coefficients and $z_0 \in \mathbb{C} \setminus \operatorname{supp}(\gamma)$. Then $\operatorname{Ind}(\gamma, z_0)$ is an integer and*

$$\operatorname{Ind}(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}.$$

Proof. We may assume that $z_0 = 0$, since $\operatorname{Ind}(\gamma, z_0) = \operatorname{Ind}(\tilde{\gamma}, 0)$, where $\tilde{\gamma}$ is γ translated by $-z_0$.

Let us first prove the formula. It is enough to show that $\operatorname{Re}\left(\int_\gamma \frac{dz}{z}\right) = 0$. Note that $\log(|z|)$ (with the usual $\log\colon (0, \infty) \to \mathbb{R}$) is a well-defined C^1 function in $\mathbb{C} \setminus \{0\}$ with

$$d[\log(|z|)] = \frac{1}{2} d[\log(|z|^2)] = \frac{\bar{z} dz + z d\bar{z}}{2|z|^2} = \operatorname{Re}\left(\frac{dz}{z}\right).$$

[2] If ω is a 1-form we denote by $\operatorname{Im}(\omega)$ the 1-form defined by $\operatorname{Im}(\omega)(v) = \operatorname{Im}(\omega(v))$ for tangent vectors v and similarly for $\operatorname{Re}(\omega)$.

[3] One can also explain the formula geometrically: Suppose that $z_0 = 0$ and that we want to measure the change in angle when we are at point z and move in the direction v. By dividing by z we scale the picture so that we are on the unit circle, and also rotate it so that z becomes 1. The tangent to the unit circle in the positive direction points up at 1, so taking the imaginary part of v/z measures the change in angle in radians to the first order.

Thus, we have

$$\mathrm{Re}\left(\int_\gamma \frac{dz}{z}\right) = \int_\gamma \mathrm{Re}\left(\frac{dz}{z}\right) = \int_\gamma d[\log(|z|)] = \int_{\partial\gamma} \log(|z|) = 0$$

since γ is closed.

Let us next show that $\mathrm{Ind}(\gamma, 0)$ is an integer. Suppose that $\gamma = c_1\gamma_1 + \dots + c_n\gamma_n$ where $c_1, \dots, c_n \in \mathbb{Z}$ and $\gamma_1, \dots, \gamma_n$ are 1-cells. We claim that

$$\exp\left(\int_{\gamma_k} \frac{dz}{z}\right) = \frac{\gamma_k(1)}{\gamma_k(0)},$$

which seems plausible, since formally $\int_{\gamma_k} \frac{dz}{z} = \log(\gamma_k(1)) - \log(\gamma_k(0))$. To show this, let

$$I(t) = \int_0^t \frac{\gamma_k'(s)}{\gamma_k(s)}\, ds$$

so that $I(1) = \int_{\gamma_k} \frac{dz}{z}$ and define also $u(t) = \gamma_k(t)e^{-I(t)}$. We have

$$u'(t) = (\gamma_k'(t) - \gamma_k(t)I'(t))e^{-I(t)} = \left(\gamma_k'(t) - \gamma_k(t)\frac{\gamma_k'(t)}{\gamma_k(t)}\right)e^{-I(t)} = 0,$$

so that u is constant and $\gamma_k(0) = u(0) = u(1) = \gamma_k(1)\exp\left(-\int_{\gamma_k} \frac{dz}{z}\right)$ as wanted. Raising this to power c_k and multiplying over k we get

$$\frac{\gamma_1(1)^{c_1} \dots \gamma_n(1)^{c_n}}{\gamma_1(0)^{c_1} \dots \gamma_n(0)^{c_n}} = e^{2\pi i\, \mathrm{Ind}(\gamma, 0)}$$

and since γ is closed, the end points of the γ_k must cancel each other and the left-hand side equals 1. This can only happen if $\mathrm{Ind}(\gamma, 0) \in \mathbb{Z}$. □

Theorem 5.5. *The function* $z \mapsto W(\gamma, z)$ *is continuous in* $\mathbb{C} \setminus \mathrm{supp}(\gamma)$*. In particular, if* γ *is closed, then* $\mathrm{Ind}(\gamma, z)$ *is constant in each connected component of* $\mathbb{C} \setminus \mathrm{supp}(\gamma)$*.*

Proof. Exercise. □

The main theorem of this section is the following. Its proof has been adapted from [1, Theorem 6.11].

Theorem 5.6. *A closed* 1*-chain in* U *is a boundary in* U *if and only if* $\mathrm{Ind}(\gamma, z) = 0$ *for all* $z \notin U$*.*

Proof. Let us first assume that γ is a boundary and $z_0 \notin U$. Then $(z - z_0)^{-1}$ is holomorphic in U, and we have $\mathrm{Ind}(\gamma, z_0) = 0$ by Stokes' theorem.

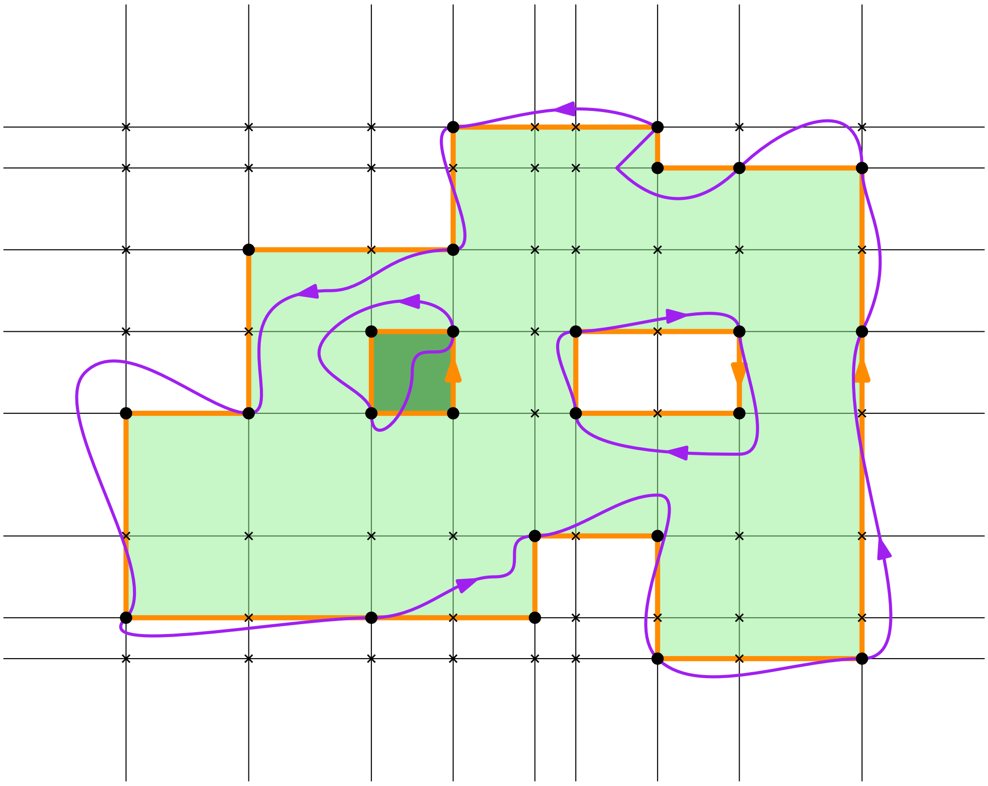

Figure 5.2: The 1-chain γ in purple and α in orange. The grid G has been drawn in thin lines. Rectangles in light green have index 1, in dark green index 2, and in white index 0 with respect to α. The round markers represent the endpoints of the segments in α before subdivision, while crosses represent the vertices of the grid.

The other direction is trickier since we need to construct a suitable 2-chain in U with γ as its boundary. Our proof will be based on a couple of lemmas.

Lemma 5.7. *There exists a* 1*-chain* α *in* U *such that* $\gamma - \alpha$ *is a boundary in* U *and each* 1*-cell in* α *is either a horizontal or a vertical line segment.*

Proof. Since sums of boundaries are boundaries, it is enough to show this for each 1-cell γ_j in γ separately. Moreover, by using uniform continuity and subdividing γ_j if needed, we may assume that $\gamma_j([0,1]) \subset B(\gamma_j(0), r/2)$ where r is the distance from γ_j to ∂U. Let α_j be the contour going first horizontally from $\gamma_j(0)$ to $(\mathrm{Re}(\gamma_j(1)), \mathrm{Im}(\gamma_j(0)))$ and then continuing vertically to Q. Then both γ_j and α_j have the same endpoints and are contained in $B(\gamma_j(0), r/2)$ which is a convex set. They are thus homotopic, implying that $\gamma_j - \alpha_j$ is a boundary in U by Theorem 3.17. □

Consider now the 1-chain α from the lemma above. Note that α is closed since γ and $\alpha - \gamma$ are, and our task has been reduced to showing that α is a boundary in U. Let G be the (finite) union of all horizontal and vertical lines that pass through one of the endpoints of the segments α_j in the chain, see Figure 5.2. Then G forms a grid such that $\operatorname{supp}(\alpha) \subset G$ and by subdividing α and ignoring constant paths we may assume that $\alpha = \sum_{j=1}^{m} c_j \alpha_j$, where $c_j \in \mathbb{Z} \setminus \{0\}$ and each α_j is a segment between adjacent vertices of the grid. Let $R_1, \ldots, R_n$ denote the closed rectangles whose interiors form the bounded connected components of $\mathbb{C} \setminus G$ and let z_j be the center point of R_j. Let $S = \sum_{j=1}^{n} \operatorname{Ind}(\alpha, z_j) R_j$.

Lemma 5.8. *Viewing R_j as 2-cells, we have $\alpha = \partial S = \sum_{j=1}^{n} \operatorname{Ind}(\alpha, z_j) \partial R_j$.*

Proof. Consider the 1-chain $\beta = \alpha - \sum_{j=1}^{n} \operatorname{Ind}(\alpha, z_j) \partial R_j$. We want to show that $\beta = 0$. Clearly $\operatorname{Ind}(\beta, z_j) = 0$ for all $1 \le j \le n$ since $\operatorname{Ind}(\partial R_k, z_j) = 1$ if $j = k$ and 0 otherwise. Moreover, we also have $\operatorname{Ind}(\beta, z) = \operatorname{Ind}(\alpha, z) = 0$ when z belongs to an unbounded connected component of $\mathbb{C} \setminus G$, since

$$\operatorname{Ind}(\alpha, z) = \frac{1}{2\pi i} \int_\alpha \frac{dw}{w - z} \to 0$$

as $|z| \to \infty$. Suppose, to obtain a contradiction, that there is some rectangle R_j with a boundary edge σ that appears in β with a coefficient $m \neq 0$. Consider $\beta' = \beta - m \partial R_j$. Clearly $\operatorname{Ind}(\beta', z) = -m$ for points in the interior of R_j, while we still have $\operatorname{Ind}(\beta', z) = 0$ for z in other connected components of $\mathbb{C} \setminus G$. However, $\operatorname{supp}(\beta')$ does not contain σ, and hence $\operatorname{Ind}(\beta', z)$ should not change when z crosses σ, giving us a contradiction. □

To finish the proof we still need to show that S is a 2-chain in U. To that end, it is enough to show that $\operatorname{Ind}(\alpha, z_j) = 0$ whenever R_j is not fully contained in U. Suppose that $z \in R_j \setminus U$ for some j. Then $\operatorname{Ind}(\alpha, z) = \operatorname{Ind}(\gamma, z) = 0$ by construction, and moreover z lies in the same connected component of $\mathbb{C} \setminus \operatorname{supp}(\alpha)$ as z_j, since either z lies in the interior of R_j, or it lies on the boundary, but the corresponding boundary segment cannot be part of $\operatorname{supp}(\alpha) \subset U$. Hence, we have $\operatorname{Ind}(\alpha, z_j) = 0$ as well. □

5.2 *Residue theorem*

The residue theorem yields a mechanical approach to doing contour integrals of functions with point-singularities.

Definition 5.9. Let $f : B(z_0, r) \setminus \{z_0\} \to \mathbb{C}$ be holomorphic with Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

We define the **residue** $\operatorname{Res}(f, z_0)$ of f at z_0 by setting $\operatorname{Res}(f, z_0) := a_{-1}$. ◆

Let us now state the residue theorem.

Theorem 5.10 (Residue theorem). *Let $U \subset \mathbb{C}$ be open and γ be a closed 1-chain in U such that $\operatorname{Ind}(\gamma, z) = 0$ for all $z \in \mathbb{C} \setminus U$. Suppose that $z_1, \dots, z_n \in U \setminus \operatorname{supp}(\gamma)$ are pairwise distinct and that $f : U \setminus \{z_1, \dots, z_n\} \to \mathbb{C}$ is holomorphic. Then*

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{k=1}^{n} \operatorname{Ind}(\gamma, z_k) \operatorname{Res}(f, z_k).$$

Proof. Since $\operatorname{Ind}(\gamma, z) = 0$ for all $z \in \mathbb{C} \setminus U$, γ is a boundary in U. Let us choose $r_1, \dots, r_n > 0$ such that $B_k = B(z_k, r_k)$ satisfies $\overline{B_k} \subset U \setminus \{z_1, \dots, z_n\} \cup \{z_k\}$ for all $1 \leq k \leq n$. Note that the 1-chain

$$\gamma - \sum_{k=1}^{n} \operatorname{Ind}(\gamma, z_k) \partial B_k$$

is a boundary in $U \setminus \{z_1, \dots, z_k\}$. Hence, by Stokes' theorem we have

$$\int_\gamma f(z)\, dz = \sum_{k=1}^{n} \operatorname{Ind}(\gamma, z_k) \int_{\partial B_k} f(z)\, dz.$$

Since the Laurent series $f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_k)^j$ around z_k converges uniformly on ∂B_k, we have

$$\int_{\partial B_k} f(z)\, dz = \sum_{j=-\infty}^{\infty} a_j \int_{\partial B_k} (z - z_k)^j\, dz = 2\pi i a_{-1} = 2\pi i \operatorname{Res}(f, z_k). \qquad \square$$

The most common situation where the residue theorem applies is when $U = \mathbb{C}$ and γ is any closed contour.

Example 5.11. Let γ be the boundary of the square $[-10, 10]^2$ and

$$f(z) = \frac{\cos(z)}{(z-1)(z-2)^2(z+100)}.$$

We want to compute

$$\int_\gamma f(z)\, dz.$$

Since f is holomorphic in $\mathbb{C} \setminus \{1, 2, -100\}$, the residue theorem applies with $U = \mathbb{C}$. Note that -100 lies outside the square, so $\operatorname{Ind}(\gamma, -100) = 0$, while $\operatorname{Ind}(\gamma, 1) = \operatorname{Ind}(\gamma, 2) = 1$. We then compute the residues. Note that since

$(z-1)f(z)$ is bounded near 1, f has a 1st order pole at 1. Hence,

$$\operatorname{Res}(f,1) = \lim_{z\to 1}(z-1)f(z) = \frac{\cos(1)}{101}.$$

The pole at 2 is of 2nd order, so if we let $g(z) = (z-2)^2 f(z)$, we have $\operatorname{Res}(f,2) = g'(2)$. Computing the derivative we have

$$g'(z) = \frac{-\sin(z)(z-1)(z+100) - (z+100+z-1)\cos(z)}{(z-1)^2(z+199)^2},$$

so

$$g'(2) = \frac{-102\sin(2) - 103\cos(2)}{201^2},$$

giving us

$$\int_\gamma f(z)\,dz = 2\pi i\left(\frac{\cos(1)}{101} + \frac{-102\sin(2) - 103\cos(2)}{201^2}\right). \qquad \blacklozenge$$

5.3 *Applications to integrals*

One application where contour integration can turn out to be useful is in evaluating integrals of analytic functions over the real line. The idea is to integrate from $-R$ to R and then close the contour by adding a curve from R back to $-R$ in such a way that the integral over the added part will tend to 0 as $R \to \infty$. The integral over the closed contour can then be evaluated using the residue theorem.

Example 5.12. Consider the integral $\int_{-\infty}^{\infty} \frac{e^{itx}}{\pi(1+x^2)}\,dx$. This is the Fourier transform of the function $(1+x^2)^{-1}/\pi$, or in probabilistic terms, the characteristic function of the Cauchy distribution.

Note first that the integral converges since the integrand is $O(x^{-2})$ as $|x| \to \infty$ and has no singularities. Moreover, since $e^{itx} = \cos(tx) + i\sin(tx)$ where cos is an even function and sin is odd, we see that the imaginary part of the integral vanishes and the value of the integral does not change if we replace t by $-t$. We can thus assume that $t \geq 0$.

Our strategy for computing the integral is as follows: We consider a closed contour γ_R that consists of the segment $[-R, R]$ and the semicircle $C_R(t) = Re^{i\pi t}$, $t \in [0,1]$. Then as $R \to \infty$ the integral $\int_{-R}^{R} \frac{e^{itz}}{\pi(1+z^2)}\,dz$ tends to the final integral we are after, so if we can show that $\int_{C_R} \frac{e^{itz}}{\pi(1+z^2)}\,dz$ tends to 0, we see that $\int_{\gamma_R} \frac{e^{itz}}{\pi(1+z^2)}\,dz \to \int_{-\infty}^{\infty} \frac{e^{itx}}{\pi(1+x^2)}\,dx$. We can then evaluate $\int_{\gamma_R} \frac{e^{itz}}{\pi(1+z^2)}\,dz$ by using the residue theorem to get the result.

Let us first show that $E_R = \int_{C_R} \frac{e^{itz}}{\pi(1+z^2)}\,dz \to 0$. By taking absolute values we see that $|E_R| \le \int_{C_R} \frac{|e^{itz}|}{\pi|1+z^2|}\,|dz|$. Note that $|e^{itz}| = e^{\mathrm{Re}(itz)} < 1$ since $t \ge 0$ and $\mathrm{Im}(z) \ge 0$ for $z \in \mathrm{supp}(C_R)$. The length of C_R is πR, and by the triangle inequality we have $|1 + z^2| \ge |z|^2 - 1 = R^2 - 1$, so that

$$E_R \le \frac{\pi R}{\pi(R^2 - 1)} \to 0$$

as $R \to \infty$.

Finally, let us compute $\int_{\gamma_R} \frac{e^{itz}}{\pi(1+z^2)}\,dz$. The function $\frac{e^{itz}}{\pi(1+z^2)}$ has first order poles at $z = \pm i$. By inspection, $\mathrm{Ind}(\gamma_R, i) = 1$ and $\mathrm{Ind}(\gamma_R, -i) = 0$. We have

$$\mathrm{Res}\left(\frac{e^{itz}}{\pi(1+z^2)}, i\right) = \lim_{z\to i}(z-i)\frac{e^{itz}}{\pi(1+z^2)} = \frac{e^{-t}}{2\pi i}.$$

Hence,

$$\int_{\gamma_R} \frac{e^{itz}}{\pi(1+z^2)}\,dz = 2\pi i\,\mathrm{Ind}(\gamma_R, i)\,\mathrm{Res}\left(\frac{e^{itz}}{\pi(1+z^2)}, i\right) = e^{-t}.$$

By symmetry we then have

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{\pi(1+x^2)}\,dx = e^{-|t|}$$

for all $t \in \mathbb{R}$. ◆

5.4 *Applications to series*

Contour integration can also help in evaluating certain series. The idea is to choose a suitable function that has the terms of the series as residues.

A common strategy is to use the function $g(z) = \pi\cot(\pi z) = \pi\frac{\cos(\pi z)}{\sin(\pi z)}$ as a building block. This function has simple poles at $z \in \mathbb{Z}$ with residue 1, as shown by the computation

$$\lim_{z\to n} \frac{(z-n)\pi\cos(\pi z)}{\sin(\pi z)} = \frac{\pi\cos(\pi n)}{\pi\sin'(\pi n)} = 1.$$

We may also compute for $z = x + iy$ that

$$\begin{aligned}|g(z)|^2 &= \left|\frac{\pi\cos(\pi z)}{\sin(\pi z)}\right|^2 = \pi^2\left|\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}}\right|^2 = \pi^2\frac{e^{2\pi y} + e^{-2\pi y} + e^{2\pi i x} + e^{-2\pi i x}}{e^{2\pi y} + e^{-2\pi y} - e^{2\pi i x} - e^{-2\pi i x}} \\ &= 1 + \frac{\cos(2\pi x)}{\cosh(2\pi y) - \cos(2\pi x)}.\end{aligned}$$

This is in particular bounded on the boundary of the square $R_n = [-n-1/2, n+1/2]^2$, uniformly in n. The idea is now to multiply g by a function that corresponds to the series we are trying to compute.

Example 5.13. Suppose that we want to compute

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2}.$$

We consider the function $h(z) = g(z)/(1+z^2)$ where $g(z) = \pi \cot(\pi z)$ and integrate over ∂R_n to get

$$\frac{1}{2\pi i} \int_{\partial R_n} h(z)\, dz = \sum_{k=-n}^{n} \operatorname{Res}(h,k) + \operatorname{Res}(h,i) + \operatorname{Res}(h,-i).$$

The residue $\operatorname{Res}(h,k)$ is simply $1/(1+k^2)$. As the integral tends to 0 as $n \to \infty$, we have

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \frac{1}{1+k^2} &= -\operatorname{Res}(h,i) - \operatorname{Res}(h,-i) = -\frac{g(i)}{2i} - \frac{g(-i)}{-2i} \\ &= \pi i \frac{\cos(\pi i)}{\sin(\pi i)} = \pi \frac{e^{\pi} + e^{-\pi}}{e^{\pi} - e^{-\pi}} = \pi \frac{\cosh(\pi)}{\sinh(\pi)} = \pi \coth(\pi).\end{aligned}$$

Thus,

$$\sum_{k=1}^{\infty} \frac{1}{1+k^2} = \frac{1}{2}(\pi \coth(\pi) - 1).$$ ◆

Let us note that for this method to work well one usually needs to sum the values of an even function. Treating odd functions can in fact be very hard. For instance, the values $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$ of the Riemann zeta function for $n \geq 2$ have a relatively simple representation if n is even, but for odd n much less is known about them.

Basic rigidity theorems

'Rigidity' in mathematics is a loose term describing phenomena where something is completely determined by less information than one might expect to be needed to specify the object in the first place. In the context of complex analysis a common example is that a given holomorphic function is constant because it satisfies some weaker-looking condition.

6.1 *The maximum principle*

Cauchy's integral formula can be written in a slightly different way to show that holomorphic functions satisfy the **mean value property**: The value of the function at a given point is equal to its average over a circle around that point.

Theorem 6.1 (Mean value theorem)**.** *Let f be a holomorphic function in U. Then for any $z \in U$ and $r > 0$ such that $\overline{B}(z,r) \subset U$ we have*

$$f(z) = \int_0^1 f(z + re^{2\pi i t})\, dt.$$

Proof. By Cauchy's integral formula we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\partial B(z,r)} \frac{f(w)}{w-z}\, dw = \frac{1}{2\pi i}\int_0^1 \frac{f(z+re^{2\pi i t})}{re^{2\pi i t}} 2\pi i r e^{2\pi i t}\, dt \\ &= \int_0^1 f(z_0 + re^{2\pi i t})\, dt. \end{aligned}$$

□

Remark. Notice that this implies in particular that if $f = u + iv$ is holomorphic then also its real and imaginary parts u and v have the **mean value property** $u(z) = \int_0^1 u(z + re^{2\pi i t})\, dt$. It turns out (but we won't prove) that a function u satisfies the mean value property if and only it is **harmonic**, meaning that $\Delta u = 0$ where $\Delta u \coloneqq \partial_x^2 u + \partial_y^2 u = 4\partial_z \partial_{\overline{z}} u$.

Conversely, one can show that a given harmonic $u\colon U \to \mathbb{R}$ in a simply connected open set U has a **harmonic conjugate** $v\colon U \to \mathbb{R}$ such that $u + iv$ is holomorphic. Indeed, one can define v by

$$v(w) = i\int_{\gamma_w} \partial_{\overline{z}} u(z)\, d\overline{z},$$

where γ_w is an arbitrary contour from a fixed point $w_0 \in U$ to $w \in U$. It is easy to check that the integral does not depend on the choice of γ_w since U is simply connected and $\partial_z \partial_{\bar{z}} u = 0$. One then has $\partial_{\overline{w}} v(w) = i\partial_{\overline{w}} u(w)$ which implies that $f = u + iv$ satisfies $\partial_{\bar{z}} f(z) = 0$. ◆

From the mean value theorem one can derive the following surprisingly powerful theorem (see also Theorem 6.10 for a generalization).

Theorem 6.2 (Maximum principle for circles). *Let f be holomorphic in U and suppose that $z \in U$ and $r > 0$ are such that $\overline{B}(z, r) \subset U$. Then*

$$|f(z)| \leq \sup_{w \in \overline{B}(z,r)} |f(w)|$$

with equality if and only if f is a constant in $\overline{B}(z, r)$.

Before the proof let us check when equality holds in the triangle inequality for integrals.

Lemma 6.3 (Triangle inequality). *Suppose that $f : [0, 1] \to \mathbb{C}$ is continuous. Then $|\int_0^1 f(t)\, dt| \leq \int_0^1 |f(t)|\, dt$ and the equality holds if and only if f is of the form $f(t) = r(t)e^{i\theta}$ for some $r : [0, 1] \to [0, \infty)$ and $\theta \in [0, 2\pi)$.*

Proof. Recall that integration is strictly increasing in the following sense: If $g, h : [0, 1] \to \mathbb{R}$ are continuous real-valued functions such that $g(t) \leq h(t)$ for all $t \in [0, 1]$, then

$$\int_0^1 g(t)\, dt \leq \int_0^1 h(t)\, dt$$

with equality if and only if $g(t) = h(t)$ for all $t \in [0, 1]$.

Let now $f : [0, 1] \to \mathbb{C}$ be continuous and choose $\theta \in [0, 2\pi)$ so that $e^{-i\theta} \int_0^1 f(t)\, dt \in [0, \infty)$ and let $g(t) = \mathrm{Re}(e^{-i\theta} f(t))$ and $h(t) = |f(t)|$. Then we have

$$\left|\int_0^1 f(t)\, dt\right| = e^{-i\theta} \int_0^1 f(t)\, dt = \int_0^1 g(t)\, dt \leq \int_0^1 h(t)\, dt = \int_0^1 |f(t)|\, dt.$$

The inequality is an equality if and only if $\mathrm{Re}(e^{-i\theta} f(t)) = |f(t)|$ for all $t \in [0, 1]$, which can only happen if $e^{-i\theta} f(t) = |f(t)|$. □

Proof of Theorem 6.2. By the triangle inequality we have

$$|f(z)| = \left|\int_0^1 f(z + re^{2\pi it})\, dt\right| \leq \int_0^1 |f(z + re^{2\pi it})|\, dt \leq \sup_{w \in \partial B(z,r)} |f(w)|.$$

The first inequality is an equality if and only if there exists a constant $\theta \in [0, 2\pi)$ such that $f(w) = |f(w)|e^{i\theta}$ for all $w \in \partial B(z, r)$, while the second inequality is an equality if and only if $|f(w)|$ is constant. Hence, $|f(z)| = \sup_{w\in\partial B(z,r)} |f(w)|$ implies that $f(w)$ is constant on $\partial B(z,r)$ and by Cauchy's integral formula we see that f is constant in $B(z,r)$. □

As an application, let us prove the fundamental theorem of algebra.

Theorem 6.4 (Fundamental theorem of algebra)**.** *Let* $p(z) = a_n z^n + \cdots + a_1 z + a_0$ *be a polynomial of degree* $n \geq 1$*. Then there exists* $z_0 \in \mathbb{C}$ *such that* $p(z_0) = 0$.

Proof. Suppose that the claim is not true, i.e. there exists a polynomial p for which $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then also $f(z) = 1/p(z)$ is a holomorphic function on $\mathbb{C}$. As $|p(z)| \geq \frac{|a_n|}{2}|z|^n$ for $|z|$ large enough[1], we have that $\sup_{z\in\partial B(0,R)} |f(z)| \to 0$ as $R \to \infty$. In particular there exists $R > 0$ such that $\sup_{z\in\partial B(0,R)} |f(z)| < |f(0)|$, which contradicts the maximum principle. □

Remark. Recall that if r is a root of a polynomial p, then $p(z) = (z-r)q(z)$ for some polynomial q. From Theorem 6.4 it then follows that every complex polynomial $p(z) = a_n z^n + \cdots + a_1 z + a_0$ can be written in the form

$$p(z) = a_n(z - r_1)\dots(z - r_n),$$

where $r_1, \dots, r_n$ are the roots of p (some roots might appear multiple times). ◆

6.2 *Liouville's theorem*

Liouville's theorem is somewhat similar to the maximum principle in that it allows one to show that a given function is constant by looking at its size, but it only works when the function is defined and analytic on the whole complex plane. Holomorphic functions $\mathbb{C} \to \mathbb{C}$ are sometimes called **entire**.

Theorem 6.5 (Liouville's theorem)**.** *Let* $f : \mathbb{C} \to \mathbb{C}$ *be a bounded holomorphic function. Then* f *is constant.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the Taylor series of f at 0, which converges everywhere by assumption. Then the function $g(z) = f(1/z)$ has the Laurent series

$$\sum_{n=-\infty}^{0} a_{-n} z^{-n}$$

converging in $\mathbb{C}\setminus\{0\}$. By assumption g is bounded near 0, so it has a removable singularity at 0 and hence $a_n = 0$ for all $n \geq 1$. □

[1] Note for instance that if $M = \max_{1\leq k\leq n-1} |a_k|$, then for $|z| > \max(1, \frac{2nM}{|a_n|})$ we have $|p(z)| \geq |a_n||z^n| - \sum_{k=0}^{n-1} |a_k||z|^k \geq |a_n||z^n| - nM|z|^{n-1} = \frac{|a_n|}{2}|z|^n + |z|^{n-1}\left(\frac{|a_n|}{2}|z| - nM\right) \geq \frac{|a_n|}{2}|z|^n$.

Example 6.6. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and non-constant. We claim that the image of f is dense. Suppose that this is not the case. Then there exists a ball $B(z_0, r) \subset \mathbb{C} \setminus f(\mathbb{C})$. The function

$$\frac{1}{f(z) - z_0}$$

is therefore bounded, so that by Liouville's theorem it is constant. It easily follows that $f(z)$ is constant as well. ◆

Remark. Liouville's theorem can in fact be strengthened quite a lot. A non-constant holomorphic function $\mathbb{C} \to \mathbb{C}$ has to take as values all complex numbers, except at most one. This result is known as the Little Picard Theorem, but its proof is slightly outside the scope of this course. The exponential function does not take 0 as a value and serves as an example that one cannot improve on the result.

Another extension of Liouville's theorem says that if $|f(z)|$ grows at most polynomially as $|z| \to \infty$, then f has to be a polynomial. We leave its proof as an exercise. ◆

6.3 *The identity theorem*

The identity theorem says that if the zeros of an analytic function have an accumulation point inside the domain then the function has to be 0.

Theorem 6.7 (Identity theorem)**.** *Suppose that U is connected and let $f : U \to \mathbb{C}$ be a holomorphic function. Suppose that there exists $z_0 \in U$ and a sequence $z_n \in U \setminus \{z_0\}$ such that $z_n \to z_0$ and $f(z_n) = 0$ for every $n \geq 1$. Then $f(z) = 0$ for all $z \in U$.*

Proof. Consider the set

$$Z = \{z \in U : \exists (z_n)_{n=1}^{\infty} \subset U \setminus \{z\} \text{ with } z_n \to z \text{ and } f(z_n) = 0 \text{ for all } n \geq 1\}.$$

Clearly $z_0 \in Z$ so Z is nonempty, and if $z \in Z$ then $f(z) = 0$ by continuity. The set Z is closed in the subspace topology of U, since if $z_n \in Z$ converge to some $z \in U$, then either the sequence is eventually constant and hence $z \in Z$, or there exists a subsequence $z_{n_k} \neq z$ with $f(z_{n_k}) = 0$, implying $z \in Z$. If we can show that Z is open, we will be done since U is connected.

Suppose thus that $z_0 \in Z$ and consider a ball $B(z_0, r) \subset U$. If $f(z) = 0$ for all $z \in B(z_0, r)$, then $B(z_0, r) \subset Z$ and we are done. On the other hand, if $f(z) \neq 0$ for some $z \in B(z_0, r)$, then the power series of f around z_0 cannot be identically 0. This means that f has a zero of order n at z_0 for some finite $n \geq 1$. Let $h(z) = f(z)/(z - z_0)^n$. Then $\lim_{z \to z_0} h(z)$ exists and is nonzero by definition, but on the other hand since $z_0 \in Z$, there exists a sequence

$z_n \in B(z_0, r) \setminus \{z_0\}$ such that $f(z_n) = 0$, implying that $h(z_n) = 0$ as well and thus $h(z_0) = \lim_{n\to\infty} h(z_n) = 0$, which is a contradiction. □

Corollary 6.8. *Suppose that $f, g\colon U \to \mathbb{C}$ are holomorphic functions on a connected open set $U \subset \mathbb{C}$ and suppose that the set $\{z \in U : f(z) = g(z)\}$ contains an accumulation point. Then the two functions must coincide on U.*

Proof. Apply the identity theorem to $f - g$. □

Example 6.9. Suppose that we know that $f: \mathbb{C} \to \mathbb{C}$ is holomorphic and $f(x) = e^x(x^2 + 2x)$ when $x \in \mathbb{R}$. Then the same formula must be true also for $x \in \mathbb{C}$. This can be used to cheaply extend some familiar formulas from real to complex setting. For instance, since

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1$$

holds for $x \in \mathbb{R}$ and both sides are analytic, the same has to hold also for $x \in \mathbb{C}$. ◆

Using the identity theorem we can prove the following general maximum principle.

Theorem 6.10 (Maximum principle)**.** *Let $U \subset \mathbb{C}$ be a connected open set and let $f : U \to \mathbb{C}$ be holomorphic. If $|f|$ has a local maximum at some point $z_0 \in U$, then f is constant.*

Proof. Suppose that f has a local maximum at x_0. Then there exists $r > 0$ such that $\overline{B}(z_0, r) \subset U$ and $|f(z)| \leq |f(a)|$ for $z \in \overline{B}(z_0, r)$. Then by Theorem 6.2 we see that f is constant in $B(z_0, r)$ and by the identity theorem it has to be constant everywhere in U. □

Corollary 6.11. *Suppose that $U \subset \mathbb{C}$ is a bounded connected open set and $f : U \to \mathbb{C}$ is holomorphic and extends continuously to ∂U. Then $|f(z)| \leq \sup_{w\in\partial U} |f(w)|$ for all $z \in U$ and if equality holds, then f is constant in U.*

Let us finish this section by proving the Schwarz lemma, which uses the maximum principle to improve bounds on the modulus of holomorphic functions fixing the origin.

Lemma 6.12 (Schwarz lemma)**.** *Let $f : B(0, 1) \to \mathbb{C}$ be holomorphic and suppose that $f(0) = 0$ and $|f(z)| \leq 1$ for $z \in B(0, 1)$. Then $|f(z)| \leq |z|$ for all $z \in B(0, 1)$ and $|f'(0)| \leq 1$. Moreover, if either $|f'(0)| = 1$ or $|f(z_0)| = |z_0|$ for any single point $z_0 \in B(0, 1) \setminus \{0\}$, then there exists $\theta \in [0, 2\pi)$ such that $f(z) = e^{i\theta} z$ for all $z \in B(0, 1)$.*

Proof. Note that since $f(0) = 0$, the function $g\colon B(0,1) \to \mathbb{C}$ given by

$$g(z) = \begin{cases} \frac{f(z)}{z}, & \text{if } z \neq 0 \\ f'(z), & \text{if } z = 0 \end{cases}$$

is holomorphic. By the maximum principle we thus have

$$|g(z)| \leq \sup_{|w|=r} |g(w)| \leq \frac{1}{r}$$

for any $z \in B(0,1)$ and $r \in (|z|, 1)$. Letting $r \to 1$ we get $|g(z)| \leq 1$ for all $z \in B(0,1)$, or equivalently $|f(z)| \leq |z|$ for $z \neq 0$ and $|f'(0)| \leq 1$ when $z = 0$. If either of these inequalities is an equality, then $|g(z)| = 1$ for some $z \in B(0,1)$, and again by maximum principle $g(z)$ has to be a constant of modulus 1, yielding $f(z) = e^{i\theta} z$ for some $\theta \in [0, 2\pi)$. □

6.4 *Local behavior of holomorphic maps*

In this section we will show that a holomorphic function looks locally like a power map.

Theorem 6.13. *Let $U \subset \mathbb{C}$ be open and simply connected and suppose that $f\colon U \to \mathbb{C} \setminus \{0\}$ is a holomorphic function with no zeros. Then there exists a holomorphic branch of $\log \circ f$ in U, i.e. a function $L\colon U \to \mathbb{C}$ such that $\exp(L(z)) = f(z)$ for all $z \in U$.*

Proof. Fix $z_0 \in U$ and let D be either $\mathbb{C} \setminus (-\infty, 0]$ or $\mathbb{C} \setminus [0, \infty)$, chosen so that $f(z_0) \in D$. Define $g\colon D \to \mathbb{C}$ by $g(z) = \mathrm{Log}(z)$ in the first case and by $g(z) = \mathrm{Log}_{(0,2\pi)}(z) = i\pi + \mathrm{Log}(-z)$ in the second case. In either case we have $\exp(g(z)) = z$ for $z \in D$, so g is a branch of the logarithm. Define also

$$L(z) = g(f(z_0)) + \int_{\gamma_z} \frac{f'(w)}{f(w)}\, dw,$$

where γ_z is any contour in U starting at z_0 and ending at z. The definition of L does not depend on the choice of contour as U is simply connected and all such contours are homotopic. It is also easy to check that L is holomorphic in U with $L'(z) = \frac{f'(z)}{f(z)}$. Indeed,

$$\begin{aligned} L(z+h) &= L(z) + \int_z^{z+h} \frac{f'(w)}{f(w)}\, dw = L(z) + \int_z^{z+h} \left(\frac{f'(z)}{f(z)} + o(1) \right) dw \\ &= L(z) + \frac{f'(z)}{f(z)} h + o(h) \end{aligned}$$

as $h \to 0$. Let $B(z_0, r)$ be a small enough ball such that $f(B(z_0, r)) \subset D$. In $B(z_0, r)$ the function $g \circ f$ is an antiderivative of f'/f, and we have

$$L(z) = g(f(z_0)) + \int_{z_0}^{z} \frac{f'(w)}{f(w)}\, dw = g(f(z_0)) + g(f(z)) - g(f(z_0)) = g(f(z)).$$

In particular $e^{L(z)} = f(z)$ holds for $z \in B(z_0, r)$. By the identity theorem we must have $e^{L(z)} = f(z)$ everywhere in U. □

Corollary 6.14. *Suppose that $U \subset \mathbb{C}$ is open and simply connected and $f : U \to \mathbb{C} \setminus \{0\}$ is holomorphic. Then there exists a branch of the nth root of f, i.e. a holomorphic function $h\colon U \to \mathbb{C}$ such that $h(z)^n = f(z)$ for $z \in U$. Indeed, one can choose $h(z) = e^{L(z)/n}$.*

We are now ready for the main theorem of this section.

Theorem 6.15. *Suppose that $f : B(z_0, R) \to \mathbb{C}$ is a holomorphic function that has a zero of order $m \geq 1$ at z_0. Then there exists $0 < r \leq R$ and a holomorphic homeomorphism $\varphi\colon B(z_0, r) \to \varphi(B(z_0, r))$ with $\varphi(z_0) = 0$ and such that $f(z) = (\varphi(z))^m$ for $z \in B(z_0, r)$.*

Proof. Since f has a zero of order m at z_0, we can write $f(z) = (z - z_0)^m g(z)$ for some holomorphic function $g\colon B(z_0, R) \to \mathbb{C}$ with $g(z_0) \neq 0$. In particular, there exists $r_1 > 0$ such that $U = g(B(z_0, r_1))$ does not contain 0. Let $h(z) = g(z)^{1/m}$ be a branch of mth root of g we obtained in Corollary 6.14. Then $f(z) = (\varphi(z))^m$, where $\varphi(z) = (z - z_0)h(z)$. Note that $\varphi'(z_0) = h(z_0) = g(z_0)^{1/m} \neq 0$. Hence, by the inverse function theorem φ maps $B(z_0, r)$ bijectively to an open set $\varphi(B(z_0, r))$ for some $r < r_1$ and φ^{-1} is holomorphic with $[\varphi^{-1}]'(z) = \frac{1}{\varphi'(\varphi^{-1}(z))}$. □

Corollary 6.16. *Under the assumptions of the theorem, f is m-to-1 for $z \in \varphi^{-1}(B(z_0, \varepsilon)) \setminus \{z_0\}$ for $\varepsilon > 0$ small enough so that $B(z_0, \varepsilon) \subset \varphi(B(z_0, r))$ (exercise). In particular, if a holomorphic function $g\colon U \to \mathbb{C}$ satisfies $g'(z) = 0$ at some $z \in U$, then g cannot be injective on U (note that $g(w) - g(z)$ has a zero of order at least 2 so $g(w)$ is at least 2-to-1 near z).*

Corollary 6.17 (Open mapping theorem)**.** *Let $f : U \to \mathbb{C}$ be holomorphic and non-constant. Then f is an open map, meaning that if $V \subset U$ is open, then $f(V)$ is open.*

Proof. It is enough to show that for any $z_0 \in U$ there exists a ball $B(z_0, r) \subset U$ such that $f(B(z_0, r))$ is open. Let $g(z) = f(z) - f(z_0)$. Since f is non-constant, g has a zero of order $1 \leq m < \infty$ at z_0. By Theorem 6.15, there exists $r > 0$ and a holomorphic homeomorphism $\varphi\colon B(z_0, r) \to \varphi(B(z_0, r))$ such that $g(z) = (\varphi(z))^m$ for $z \in B(z_0, r)$. Since both φ and the power map $z \mapsto z^m$ are open, $g(B(z_0, r))$ is open as well, and as translations are open, so is $f(B(z_0, r))$. □

Conformal bijections

7.1 *The Riemann sphere*

Consider the function $\iota(z) = 1/z$ defined on $\mathbb{C} \setminus \{0\}$. Although the limit as $z \to 0$ does not exist in $\mathbb{C}$, the function still behaves in a controlled manner near 0. Namely, for any $R > 0$ we have $1/z \notin B(0, R)$ once $|z|$ is small enough. Thus, $\iota(z)$ eventually escapes every bounded set of $\mathbb{C}$ as $z \to 0$, and in that sense we could say that $\iota(z) \to \infty$.

Definition 7.1. The **extended complex plane**, also known as the **Riemann sphere**, is the set $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ where a new point ∞ has been added to $\mathbb{C}$. ◆

With this definition, we may extend ι to a bijection $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by setting $\iota(0) = \infty$ and $\iota(\infty) = 0$. Note that ι is an involution[1] that maps $\hat{\mathbb{C}} \setminus \{\infty\} \leftrightarrow \hat{\mathbb{C}} \setminus \{0\}$. The set $\hat{\mathbb{C}} \setminus \{0\}$ can thus be viewed via ι as another copy of $\mathbb{C}$ and given a topology by saying that $U \subset \hat{\mathbb{C}} \setminus \{0\}$ is open if and only if $\iota^{-1}(U)$ is open in $\mathbb{C}$. Finally, we give the whole $\hat{\mathbb{C}}$ a topology by saying that a set $U \subset \hat{\mathbb{C}}$ is open if and only if $U \setminus \{\infty\}$ is open in $\hat{\mathbb{C}} \setminus \{\infty\}$ and $U \setminus \{0\}$ is open in $\hat{\mathbb{C}} \setminus \{0\}$.

Lemma 7.2. *The above construction defines a topology on* $\hat{\mathbb{C}}$.

Proof. Clearly $\varnothing$ and $\hat{\mathbb{C}}$ are open.

If U and V are open in $\hat{\mathbb{C}}$, then $U \cap V \setminus \{\infty\} = (U \setminus \{\infty\}) \cap (V \setminus \{\infty\})$ is open in $\hat{\mathbb{C}} \setminus \{\infty\}$ and similarly $U \cap V \setminus \{0\}$ is open in $\hat{\mathbb{C}} \setminus \{0\}$, so that $U \cap V$ is open in $\hat{\mathbb{C}}$.

If $(U_i)_{i \in I}$ is an arbitrary collection of open subsets of $\hat{\mathbb{C}}$, then $\bigcup_{i \in I} U_i \setminus \{\infty\} = \bigcup_{i \in I} (U_i \setminus \{\infty\})$ is open in $\hat{\mathbb{C}} \setminus \{\infty\}$ and similarly $\bigcup_{i \in I} U_i \setminus \{0\}$ is open in $\hat{\mathbb{C}} \setminus \{0\}$, showing that $\bigcup_{i \in I} U_i$ is open in $\hat{\mathbb{C}}$. □

Briefly speaking, whenever we need to study what happens near ∞, we can via the map ι instead look at the situation near 0. Thus, since $B(0, r)$, $r > 0$, form a neighborhood basis of 0, the sets $\iota(B(0, r)) = \{z \in \mathbb{C} : |z| > 1/r\} \cup \{\infty\}$ form a neighborhood basis of ∞, and in particular for a sequence $z_n \in \hat{\mathbb{C}}$ we have $z_n \to \infty \Leftrightarrow \iota(z_n) \to 0 \Leftrightarrow 1/|z_n| \to 0 \Leftrightarrow |z_n| \to +\infty$, where we have defined $|\infty| = +\infty$, the positive infinity in the extended real numbers $\mathbb{R} \cup \{\pm\infty\}$. (Note that here ∞ and $+\infty$ are not the same thing. If we have a sequence of real numbers x_n that tends to $\pm\infty$, then on $\hat{\mathbb{C}}$ they tend to ∞, but the converse is not true.)

[1] An **involution** is a function that is its own inverse.

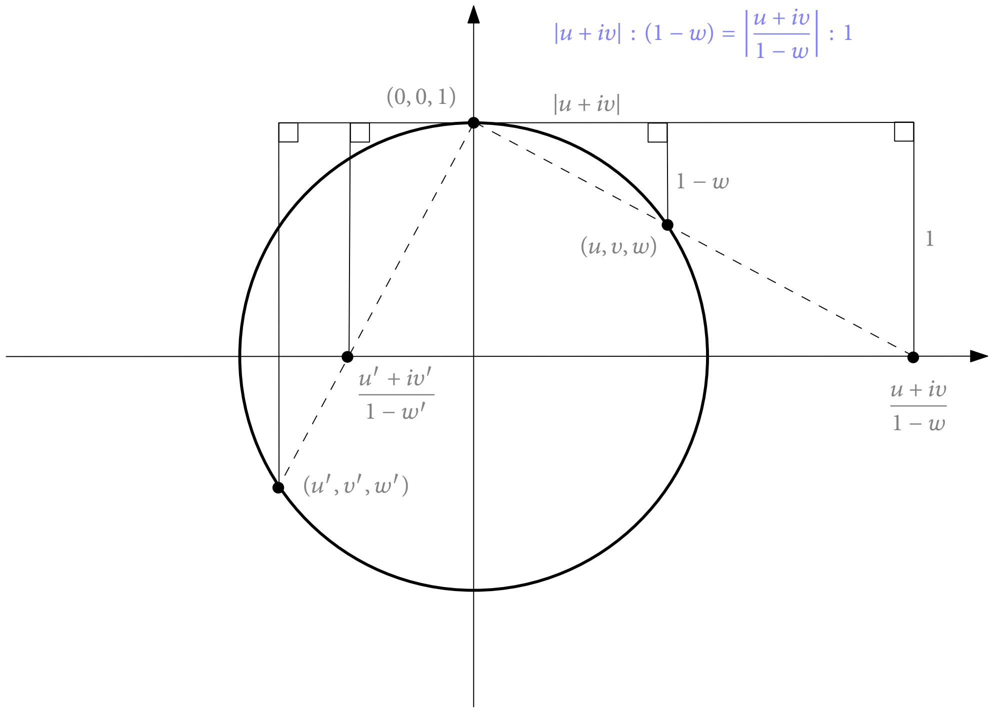

Figure 7.1: Stereographic projection viewed from the side.

Example 7.3. The functions $z \mapsto z$, $z \mapsto z^2$, $z \mapsto 1/z$ can be defined continuously on $\hat{\mathbb{C}}$ by taking limits.

On the other hand, the function $\exp(z)$ cannot be continuously extended to a function $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, since $\exp(z) \to 0$ when $z \to -\infty$ along the real axis, while $\exp(z) \to +\infty$ when $z \to +\infty$. ◆

The reason for the name Riemann sphere is that $\hat{\mathbb{C}}$ is topologically equivalent to the two-dimensional sphere. A particularly nice homeomorphism between the two is given by the **stereographic projection** which maps a point $(u, v, w) \in \mathbb{R}^3$ with $u^2 + v^2 + w^2 = 1$ to the point $\frac{u+iv}{1-w} \in \mathbb{C}$ in the case $w \in [-1, 1)$, and to ∞ in the case $(u, v, w) = (0, 0, 1)$. The point $\frac{u+iv}{1-w}$ is where the line through $(0, 0, 1)$ and (u, v, w) intersects the plane $w = 0$, which we identify with $\mathbb{C}$, see Figure 7.1. Note that the unit circle $u^2+v^2 = 1$ stays fixed, while the points with $w < 0$ map to the unit disc $B(0, 1)$ with $(0, 0, -1)$ mapping to 0. The points with $w > 0$ map to the exterior of the unit disc, with the point $(0, 0, 1)$ mapping to the point at infinity. An interesting feature of this map is that its inverse takes both lines and circles on $\mathbb{C}$ to circles on the sphere (we will skip the proof).

Above we wrote $\hat{\mathbb{C}}$ as the union of the two sets $\hat{\mathbb{C}} \setminus \{\infty\}$ and $\hat{\mathbb{C}} \setminus \{0\}$ and noted that they can be parametrized by $\mathbb{C}$: For $z \in \hat{\mathbb{C}} \setminus \{\infty\} = \mathbb{C}$ we can simply use $\mathrm{id}(z) = z$ itself as a coordinate, while for $z \in \hat{\mathbb{C}} \setminus \{0\}$ we can use $\iota(z) = 1/z =: \zeta$ as a coordinate with $\iota(\infty) = 0$. In the intersection $\hat{\mathbb{C}} \setminus \{0, \infty\}$ we have the coordinate transformation $z \mapsto \zeta = 1/z$, which is holomorphic. This makes $\hat{\mathbb{C}}$ an example of a **complex manifold** with **coordinate charts** $(\hat{\mathbb{C}} \setminus \{\infty\}, \mathrm{id})$ and $(\hat{\mathbb{C}} \setminus \{0\}, \iota)$, but we are not going to talk about manifolds in general.

Let now $U \subset \hat{\mathbb{C}}$ be open and $f : U \to \hat{\mathbb{C}}$ be a continuous function. We say that f is holomorphic near $z_0 \in U$ if it is holomorphic in suitable local coordinates around z_0 and $w_0 = f(z_0)$. More precisely, there are the following cases:

- If $z_0, w_0 \in \mathbb{C}$, we can simply check that $f(z)$ is holomorphic near z_0.
- If $z_0 \in \mathbb{C}$ and $w_0 = \infty$, then we must use the ζ-coordinate on the image side, which corresponds to checking that the function

$$g(z) = \iota(f(z)) = \begin{cases} \frac{1}{f(z)}, & \text{if } f(z) \neq \infty \\ 0, & \text{if } f(z) = \infty \end{cases}$$

  is holomorphic near z_0.
- If $z_0 = \infty$ and $w_0 \in \mathbb{C}$, we must use the ζ-coordinate on the domain side and check that

$$g(\zeta) = f(\iota^{-1}(\zeta)) = \begin{cases} f(1/\zeta), & \text{if } \zeta \neq 0 \\ w_0, & \text{if } \zeta = 0 \end{cases}$$

  is holomorphic near 0.
- Finally, if $z_0 = w_0 = \infty$, we must use the ζ-coordinate both on the domain and the image, which corresponds to showing that the function

$$g(\zeta) = \iota(f(\iota^{-1}(\zeta))) = \begin{cases} \frac{1}{f(1/\zeta)}, & \text{if } \zeta \neq 0 \text{ and } f(1/\zeta) \neq \infty \\ 0, & \text{otherwise} \end{cases}$$

  is holomorphic near 0.

Example 7.4. The function $f(z) = z^2$ is holomorphic on $\hat{\mathbb{C}}$ when we define $f(\infty) = \infty$: It is clearly holomorphic on $\mathbb{C}$, while around $z_0 = \infty$, we must check the holomorphicity of $1/(1/\zeta)^2 = \zeta^2$ near 0, which is clearly holomorphic.

Similarly, $f(z) = 1/z$ with $f(0) = \infty$ and $f(\infty) = 0$ is holomorphic. It is clearly holomorphic on $\mathbb{C} \setminus \{0\}$ since in the usual coordinates $f(z) = 1/z$ is holomorphic. Around 0 we map to ∞, so we must use ζ-coordinates on the image side and check that $1/f(z) = 1/(1/z) = z$ is holomorphic, which it is. Finally, around ∞ we map to 0, so we must use ζ-coordinates on the domain side and check that $f(1/\zeta) = 1/(1/\zeta) = \zeta$ is holomorphic, which it is. ◆

Lemma 7.5. *Let $U \subset \mathbb{C}$ be open and connected and suppose that $f : U \to \hat{\mathbb{C}}$ is continuous and not identically ∞. Let also $P = \{z \in U : f(z) = \infty\}$. Then f is holomorphic if and only if the set P has no accumulation points in U, $f : U \setminus P \to \mathbb{C}$ is holomorphic, and every singularity point $z \in P$ is a pole.*

Proof. Suppose first that f is holomorphic and not identically ∞ on U. If P has an accumulation point $z_0 \in U$, then the function $1/f(z)$ has accumulating zeros at z_0, so that by the identity theorem $1/f(z) = 0$ everywhere[2], implying that $f(z) = \infty$ for all $z \in U$, a contradiction. Next we note that if $z_0 \in P$, then $\iota \circ f$ is holomorphic $B(z_0, r) \to \mathbb{C}$ for some $r > 0$. In particular, it has a zero of order $m \geq 0$ at z_0 and $\frac{1}{(z-z_0)^m f(z)} = g(z)$ is a holomorphic function in $B(z_0, r)$ with $g(0) \neq 0$. Hence, $1/g$ is also holomorphic near z_0 and $f(z) = \frac{1}{(z-z_0)^m g(z)}$ has a pole of order m at z_0.

Conversely, suppose that P has no accumulation points and that f is holomorphic outside P with poles at P. Then for $z_0 \in P$ with pole of order m, the function $h(z) = (z - z_0)^m f(z)$ can be extended to be analytic at z_0 with $h(z_0) \neq 0$. Hence, $\iota(f(z)) = (z-z_0)^m / h(z)$ is analytic near z_0 and by definition, f is a holomorphic $\hat{\mathbb{C}}$-valued function near z_0. □

Holomorphic functions $f : U \to \hat{\mathbb{C}}$ are also called **meromorphic** (excluding the special case when $f(z) = \infty$ for all $z \in U$). Equivalently, a function is meromorphic if it can be expressed as the ratio of two holomorphic functions $f, g : U \to \mathbb{C}$, where g is not identically 0 (it is easy to see that ratios of holomorphic functions are meromorphic, but the converse is more difficult and we skip the proof). For instance, $1/\sin(z)$ is meromorphic in $\mathbb{C}$ (it has first order poles at $z = \pi i n$, $n \in \mathbb{Z}$) but $e^{1/z}$ is not since it has an essential singularity at 0. (On the other hand, $e^{1/z}$ is meromorphic (and even holomorphic) as a function $\mathbb{C} \setminus \{0\} \to \hat{\mathbb{C}}$.)

Let us close this section with the following theorem which says that holomorphic functions defined on the whole Riemann sphere are a fairly restricted class.

Theorem 7.6. *A map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is holomorphic if and only if it is a rational function (extended continuously to $\hat{\mathbb{C}}$) or $f(z) = \infty$ everywhere.*

Proof. Rational functions are clearly holomorphic since they have no essential singularities. Suppose that $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is holomorphic and not ∞ everywhere. Then f can only have finitely many points $z_1, \ldots, z_n \in \mathbb{C}$ with $f(z_k) = \infty$ for all $k = 1, \ldots, n$ (otherwise, since $\hat{\mathbb{C}}$ is compact, they accumulate somewhere). Moreover, all of these points have to be poles of f. Let Q_k be the principal part of the Laurent series of f at z_k. Also since f is holomorphic at ∞, we know

[2] It is easy to check that the identity theorem holds also for holomorphic functions that can take the value ∞.

that in the ζ coordinates $f(1/\zeta)$ has no essential singularity around 0. Let $P(\zeta)$ be the principal part of the Laurent series of $f(1/\zeta)$ at 0 and define

$$g(z) = f(z) - \sum_{k=1}^{n} Q_k(z) - P(1/z)$$

for $z \in \mathbb{C} \setminus \{z_1, \dots, z_n\}$. Then g is bounded near every z_k and also as $z \to \infty$, meaning that $g \colon \mathbb{C} \to \mathbb{C}$ is a bounded holomorphic function. Hence, by Liouville's theorem $g = c$ is constant and $f(z) = c + \sum_{k=1}^{n} Q_k(z) + P(1/z)$ is a rational function. □

7.2 *Möbius transformations*

Definition 7.7. Let $a, b, c, d \in \mathbb{C}$ be such that $ad - bc \neq 0$. A map $f \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$f(z) = \frac{az + b}{cz + d}$$

is called a **Möbius transformation**. ◆

In the definition above it is understood that if $c = 0$ then $f(\infty) = \infty$, while otherwise $f(\infty) = \frac{a}{c}$ and $f(\frac{-d}{c}) = \infty$. The condition $ad - bc \neq 0$ ensures that the map is well-defined and non-constant.

The following theorem identifies Möbius transformations as the holomorphic automorphisms of the sphere.

Theorem 7.8. *Every holomorphic bijection $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a Möbius transformation and vice versa. They form a group* $\mathrm{Aut}(\hat{\mathbb{C}})$ *under composition of mappings and* $\mathrm{Aut}(\hat{\mathbb{C}})$ *is generated by Möbius transformations of the form $z \mapsto z + a$ (translations, $a \in \mathbb{C}$), $z \mapsto \lambda z$ (rotations and scaling, $\lambda \in \mathbb{C} \setminus \{0\}$) and $z \mapsto 1/z$.*

Proof. Let us first show that every holomorphic bijection $f \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a Möbius transformation. By Theorem 7.6 f is a rational function and we can thus write

$$f(z) = \lambda \frac{(z - z_1)^{k_1} \dots (z - z_n)^{k_n}}{(z - w_1)^{\ell_1} \dots (z - w_m)^{\ell_m}}$$

for some $a, z_1, \dots, z_n, w_1, \dots, w_m, \lambda \in \mathbb{C}$ with $n, m \geq 0$, $k_j, \ell_j \geq 1$ and with $z_j \neq w_k$ for all j, k. Since f is a bijection we must have $\lambda \neq 0$. Note also that if $n \geq 2$, then f takes the value 0 at least twice and is not a bijection. Similarly, if $m \geq 2$, then f takes the value ∞ at least twice. We cannot have $n = m = 0$, since then f is a constant map. Moreover, we must have $k_j = 1$ for every $j = 1, \dots, n$, since otherwise we have $f'(z_j) = 0$, which implies that f is not injective around z_j. Similarly, we must have $\ell_j = 1$ because otherwise $1/f$ will not be injective, so neither is f. If $n = m = 1$, then $f(z) = \frac{\lambda z - \lambda z_1}{z - w_1}$ is a Möbius transformation because $z_1 \neq w_1$. If $n = 1, m = 0$, then $f(z) = \lambda(z - z_1)$ and if $n = 0, m = 1$, then $f(z) = \frac{\lambda}{z - w_1}$. In both cases f is a Möbius transformation.

Let us next show that a given Möbius transformation $f(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$ can be expressed as a composition of the given generators. Since the generators are holomorphic bijections, this also shows that f is a holomorphic bijection.

Let us first consider the case $c = 0$. Note that then a and d have to be nonzero since otherwise $ad - bc = 0$. We can use the following steps to reduce f to the identity transformation by using the generators

$$\frac{az+b}{d} \xrightarrow{z - \frac{b}{d}} \frac{az}{d} \xrightarrow{\frac{d}{a}z} z.$$

Letting $f_1(z) = z - b/d$ and $f_2(z) = dz/a$, we have $f_1^{-1}(z) = z + b/d$ and $f_2^{-1}(z) = az/d$. Thus, $\frac{az+b}{d} = (f_1^{-1} \circ f_2^{-1})(z)$ expresses $\frac{az+b}{d}$ as a composition of generators.

Suppose then that $c \neq 0$. This time we can use for instance the following steps:

$$\frac{az+b}{cz+d} \xrightarrow{z - \frac{a}{c}} \frac{bc - ad}{c^2 z + cd} \xrightarrow{\frac{z}{bc-ad}} \frac{1}{c^2 z + cd} \xrightarrow{\frac{1}{z}} c^2 z + cd \xrightarrow{z - cd} c^2 z \xrightarrow{c^{-2} z} z.$$

Again, inverting the chain lets us express f using the generators. □

Let us next discuss the degrees of freedom in Möbius transformations. It turns out that they are uniquely determined by specifying 3 points and their images.

Theorem 7.9. *Suppose that $z_1, z_2, z_3, w_1, w_2, w_3 \in \hat{\mathbb{C}}$ are six points such that $z_j \neq z_k$ and $w_j \neq w_k$ for $j \neq k$. Then there exists a unique Möbius transformation $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $f(z_k) = w_k$ for all $k = 1, 2, 3$.*

Proof. We note that the map

$$f_{z_1,z_2,z_3}(z) = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$

satisfies $f(z_1) = 0$, $f(z_2) = 1$ and $f(z_3) = \infty$. Hence, $g = f^{-1}_{w_1,w_2,w_3} \circ f_{z_1,z_2,z_3}$ is the map we are after, and it remains to show the uniqueness.

Suppose first that $f(z) = \frac{az+b}{cz+d}$ satisfies $f(z_k) = z_k$ for $k = 1, 2, 3$. We claim that f has to be the identity transform. Suppose first that none of the points z_k is ∞. Then

$$f(z) - z = \frac{az+b}{cz+d} - z = \frac{-cz^2 + (a-d)z + b}{cz+d}$$

is a rational function that has 3 zeros at z_1, z_2 and z_3. Its numerator is of degree at most 2, so this implies that the numerator has to vanish and $f(z) = z$. If $z_3 = \infty$, then we must have $c = 0$, and we again argue that

$$f(z) - z = \frac{az + b}{d} - z$$

is a polynomial of degree at most 1 but has zeros at z_1 and z_2, so it has to be 0.

Finally, suppose that z_k and w_k are arbitrary and that f and g are two Möbius transformations mapping $f(z_k) = g(z_k) = w_k$ and $g(z_k) = w_k$. Then $g^{-1} \circ f$ maps $z_k \mapsto z_k$ for $k = 1, 2, 3$, and thus it has to be the identity map and hence $g = f$. □

Consider next a circle $|z - z_0| = |z_0|$ in the plane with center at z_0 and passing through 0. As $z_0 \to \infty$ the circle looks more and more like a line. In $\hat{\mathbb{C}}$ the limiting object is in fact still topologically a circle, just passing through ∞. It therefore makes sense to define generalized circles as either circles or lines.

Definition 7.10. A **generalized circle** is a subset of $\hat{\mathbb{C}}$ that is either a circle in $\mathbb{C}$, or $L \cup \{\infty\}$ where L is a line in $\mathbb{C}$. ◆

Theorem 7.11. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius transformation. Then f maps generalized circles to generalized circles.*

Proof. It is clear that the generators $z \mapsto z + a$ and $z \mapsto az$ map generalized circles to generalized circles, so we will be done if we can show that $z \mapsto \frac{1}{z}$ does the same.

Note that the equation of a circle $|z - a| = r$ can be after squaring and expanding written in the form

$$|z|^2 - \bar{a}z - a\bar{z} + |a|^2 - r^2 = 0.$$

More generally, if we look at an equation of the form

$$c|z|^2 - \bar{a}z - a\bar{z} + d = 0$$

with $c, d \in \mathbb{R}$ and $a \in \mathbb{C}$, then if $c \neq 0$ and $cd < |a|^2$, this represents the circle with center $\frac{a}{c}$ and radius $\frac{\sqrt{|a|^2 - cd}}{|c|}$. If $c = 0$ and $a \neq 0$, we instead get the line $\operatorname{Re}(\bar{a}z) = a_1 z_1 + a_2 z_2 = d$. Now, if $w = 1/z$ then z satisfies the equation if and only if w satisfies

$$c - \overline{aw} - aw + d|w|^2 = 0,$$

which is again an equation of the same form. Thus, $1/z$ maps generalized circles bijectively to each other. □

Recall that affine transformations like scaling, rotation and translation preserve both angles and ratios of lengths. In particular, if z_1 is a complex number that we treat as the origin and z_2, z_3 are two other numbers, then

$$\frac{z_2 - z_1}{z_3 - z_1}$$

is unchanged when we map $z_j \mapsto f(z_j)$ where $f(z_j) = az+b$ for some $a, b \in \mathbb{C}$. For general Möbius transformations this is not true anymore, but a weaker property still holds: they preserve *ratios of ratios*, also known as cross-ratios.

Definition 7.12. Let $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ be distinct points. Their **cross-ratio** $(z_1, z_2; z_3, z_4)$ is defined by setting

$$(z_1, z_2; z_3, z_4) := \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} = \frac{z_3 - z_1}{z_4 - z_1} : \frac{z_3 - z_2}{z_4 - z_2}.$$

When two of the points are equal, we define the cross-ratio as a limit when one of the points tends to the other. Similarly, if one of the points z_k is ∞, we may define the cross-ratio as the corresponding limit as $z_k \to \infty$. ◆

An alternative way of thinking about the cross-ratio is as follows: Suppose that z_1, z_2, z_3 are distinct and let $h(z) = \frac{z_3 - z_1}{z_3 - z_2} \cdot \frac{z - z_2}{z - z_1}$ be the unique Möbius transformation that maps $z_1 \mapsto \infty$, $z_2 \mapsto 0$ and $z_3 \mapsto 1$. Then $(z_1, z_2; z_3, z_4) = h(z_4)$.

Theorem 7.13. *Let $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ be distinct and let f be a Möbius transformation. Then*

$$(z_1, z_2; z_3, z_4) = (f(z_1), f(z_2); f(z_3), f(z_4)).$$

Proof. Let h be the Möbius transformation that maps $z_1 \mapsto \infty$, $z_2 \mapsto 0$ and $z_3 \mapsto 1$. Then $g = h \circ f^{-1}$ is the Möbius transformation that maps $f(z_1) \mapsto \infty$, $f(z_2) \mapsto 0$ and $f(z_3) \mapsto 1$ and

$$(z_1, z_2; z_3, z_4) = h(z_4) = g(f(z_4)) = (f(z_1), f(z_2); f(z_3), f(z_4)). \qquad \square$$

Cross-ratio preservation also yields a good way of computing the coefficients of a Möbius transformation. Suppose that we want to find the Möbius transformation f that maps $z_k \mapsto w_k$, $k = 1, 2, 3$. Then instead of writing $f(z) = \frac{az+b}{cz+d}$ and solving each of the three equations $f(z_k) = w_k$ (and say $ad - bc = 1$ to fix the normalization) for a, b, c, d, we can instead write the cross-ratio condition $(z_1, z_2; z_3, z) = (w_1, w_2; w_3, f(z))$ which holds for all $z \in \mathbb{C}$ and then solve for $f(z)$.

Example 7.14. Suppose that we want to find a Möbius transformation f that maps $1 \mapsto 3$, $2 \mapsto -i$ and $3 \mapsto 1$. Since f has to preserve cross-ratios, we must have $(1, 2; 3, z) = (3, -i; 1, f(z))$, or equivalently

$$\frac{(3-1)(z-2)}{(z-1)(3-2)} = \frac{(1-3)(f(z)+i)}{(f(z)-3)(1+i)}.$$

Simplifying we get

$$\frac{2z-4}{z-1} = \frac{-2f(z)-2i}{(f(z)-3)(1+i)}.$$

Multiplying by $f(z) - 3$ we have

$$(f(z)-3)\frac{2z-4}{z-1} = \frac{-2f(z)-2i}{1+i}.$$

Collecting factors of $f(z)$ and moving the rest to the right-hand side we have

$$f(z)\left(\frac{2z-4}{z-1} + \frac{2}{1+i}\right) = \frac{6z-12}{z-1} - \frac{2i}{1+i}.$$

Multiplying by $(z-1)(1+i)$ we get

$$f(z)((2z-4)(1+i) + 2(z-1)) = (6z-12)(1+i) - 2i(z-1)$$

and simplifying gives us

$$f(z)((4+2i)z - 6 - 4i) = (6+4i)z - 12 - 10i$$

and finally

$$f(z) = \frac{(6+4i)z - 12 - 10i}{(4+2i)z - 6 - 4i} = \frac{(3+2i)z - 6 - 5i}{(2+i)z - 3 - 2i}.$$ ◆

7.3 *Conformal maps*

Conformal maps f are maps that are locally invertible and preserve angles and orientation at every point, meaning that their derivative only rotates or scales tangent vectors – see Figure 7.2 for an example. In other words, at a given point z with tangent vector v we should have $df_z(v) = \alpha(z)v$ for some complex number $\alpha(z) \in \mathbb{C} \setminus \{0\}$, but this is equivalent with having $df_z = f'(z)\,dz$ with $f'(z) \neq 0$.

Definition 7.15. A mapping $f : U \to \mathbb{C}$ is **conformal** if it is holomorphic and $f'(z) \neq 0$ for all $z \in U$. ◆

Note that since Möbius transformations are injective, they must have nonzero derivative everywhere (assuming that we use the ζ-coordinates at the pole).

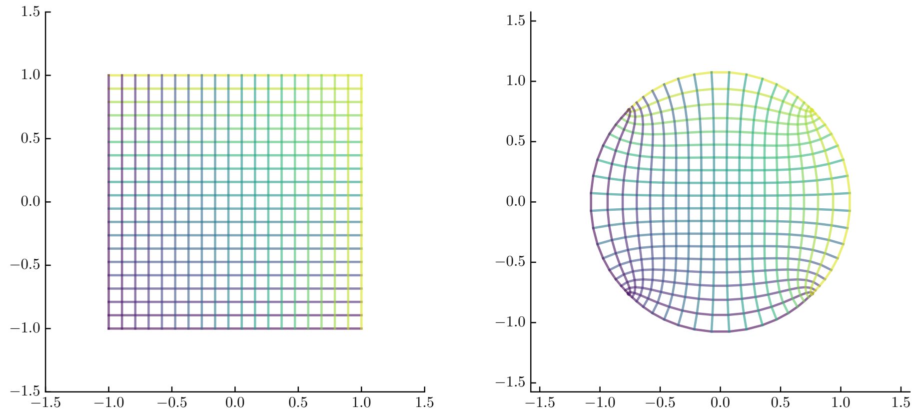

Figure 7.2: The function $f(z) = z + 0.0731647z^5 + 0.00358709z^9$ approximates a conformal map from the square $[-1,1]^2$ to the unit disc [3, p. 286] (in reality the map is given by a certain elliptic function). Note how the grid lines still intersect at right angles after the mapping except at the four points corresponding to the vertices of the square where the map is not conformal.

An important question that often arises is whether there exists a conformal bijection $U \to V$ between two open sets $U, V \subset \mathbb{C}$, and to find such a bijection if possible. Via conformal mappings it is often possible to solve a given problem in an easier domain and then transfer the results back to the original domain. The most important theorem in this direction is the Riemann mapping theorem, which we will only state here.

Theorem 7.16 (Riemann mapping theorem). *Suppose that $U \subset \mathbb{C}$ is open, nonempty, simply connected and $U \neq \mathbb{C}$. Then there exists a conformal bijection $f\colon B(0,1) \to U$.*

The conformal maps coming out of the Riemann mapping theorem are typically non-explicit, although their qualitative properties can be studied further. We will instead focus on looking at some simple concrete conformal mapping problems involving Möbius transformations and other basic functions.

Example 7.17. Let us find a Möbius transformation that takes $\partial B(0,1)$ to $\mathbb{R}$, with $B(0,1)$ mapping to the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$. We will first pick three points on $\partial B(0,1)$, say $z_1 = -1$, $z_2 = 1$ and $z_3 = i$. Next we need to decide where want to map them and pick three image points on the generalized circle $\mathbb{R} \cup \{\infty\}$. Since we want to map $B(0,1)$ to the upper half-plane, and since conformal maps preserve orientation, we can simply make sure that when we trace $\partial B(0,1)$ from z_1 to z_2 to z_3 then on the image side when we go from $f(z_1)$ to $f(z_2)$ to $f(z_3)$, the upper half-plane always stays to the left of the curve. This means that we should pick $f(z_1)$, $f(z_2)$ and $f(z_3)$ in

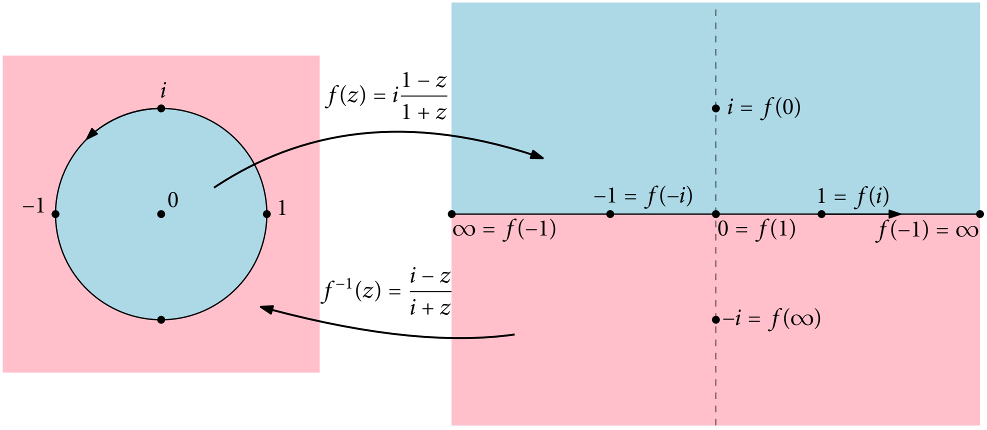

Figure 7.3: The Möbius transformation $f(z) = i\frac{1-z}{1+z}$ and its inverse.

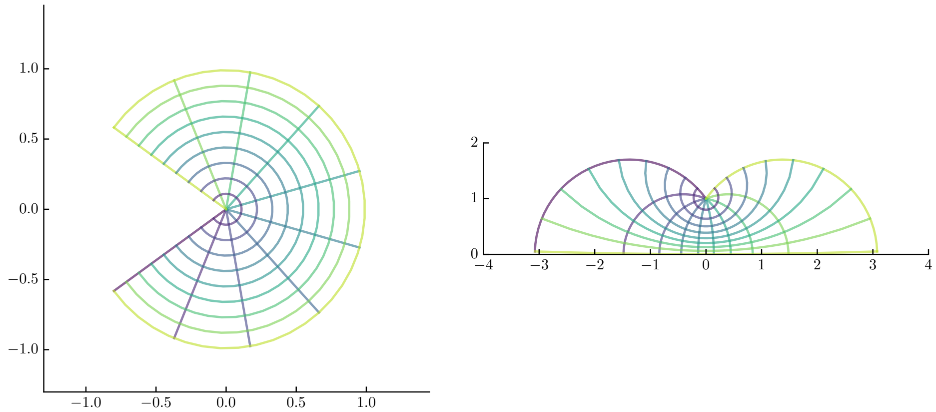

Figure 7.4: The image of Pac-Man under $f(z) = i\frac{1-z}{1+z}$ before it eats the singularity at -1.

(cyclically) increasing order. Let us pick $f(z_1) = \infty$, $f(z_2) = 0$ and $f(z_3) = 1$. In this case the map is then given by

$$f(z) = \frac{z-1}{z+1} \cdot \frac{i+1}{i-1} = i\frac{1-z}{1+z}.$$

It is easy to check that

$$f^{-1}(z) = \frac{i-z}{i+z}.$$

A schematic plot of this map and its inverse is in Figure 7.3 and the map is plotted in Figure 7.4. ◆

Example 7.18. Consider the open set $U = \{z \in \mathbb{C} : \mathrm{Re}(z), \mathrm{Im}(z) > 0\}$, which is the positive quadrant of the complex plane. It can also be viewed as the sector

$U = \{re^{i\theta} : r > 0, \theta \in (0, \pi/2)\}$. The map $f(z) = z^2$ doubles the angle and squares the radius, so it takes U bijectively to the upper half plane $\mathbb{H}$. We also have $f'(z) \neq 0$ for $z \in U$, so f is conformal. Composing this with the map in the previous example we get a conformal bijection $U \to B(0,1)$ given by

$$z \mapsto \frac{i - z^2}{i + z^2}.$$

Note that the map $z \mapsto z^2$ is not conformal at $z = 0$. Indeed, the 90° angle at the boundary of U at 0 gets transformed into a 180° angle, even though angles are preserved everywhere else. ◆

Let us finish by considering conformal maps from $\mathbb{H}$ or $B(0,1)$ to itself. One can easily find Möbius transformations that map $B(0,1) \to B(0,1)$ or $\mathbb{H} \to \mathbb{H}$ bijectively, so the main content of the theorems below is that there are no other exotic conformal bijections.

Theorem 7.19. *Let $a \in B(0,1)$ and $\theta \in [0, 2\pi)$. The Möbius transformation*

$$f(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}$$

is a conformal bijection $B(0,1) \to B(0,1)$ and all such maps are of this form.

Proof. Note first that if f is of the given form then for $z \in \partial B(0,1)$ we have

$$|f(z)| = \frac{|z - a|}{|1 - \bar{a}z|} = \frac{|z - a|}{|1 - a/z|} = \frac{|z - a|}{|z - a|} = 1$$

since $\bar{z} = 1/z$ when $|z| = 1$. Thus, f maps the unit circle to unit circle, and since $f(a) = 0$, it must map the inside of the circle to the inside of the circle.

Suppose next that $f : B(0,1) \to B(0,1)$ is a conformal bijection and let $b = f(0)$. If $\varphi(z) = \frac{z-b}{1-\bar{b}z}$, then $g = \varphi \circ f$ is a conformal bijection $B(0,1) \to B(0,1)$ with $g(0) = 0$. By Schwarz lemma we must have $|g'(0)| \leq 1$. Similarly, g^{-1} is a conformal bijection and by Schwarz lemma also $|(g^{-1})'(z)| \leq 1$. Noting that $(g^{-1})'(0) = 1/g'(0)$ we get $|g'(0)| = 1$. But then, again by Schwarz lemma, we must have $g(z) = e^{i\theta} z$ for some $\theta \in [0, 2\pi)$. Noting that $\varphi^{-1}(z) = \frac{z+b}{1+\bar{b}z}$ we have

$$f(z) = \varphi^{-1}(g(z)) = \frac{e^{i\theta} z + b}{1 + \bar{b}e^{i\theta} z} = e^{i\theta} \frac{z + e^{-i\theta} b}{1 + e^{i\theta} \bar{b} z},$$

which is of the wanted form with $a = -e^{-i\theta} b$. □

Theorem 7.20. *The conformal maps $f : \mathbb{H} \to \mathbb{H}$ are the Möbius transformations $\frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ satisfying $ad - bc > 0$.*

Proof. Let $\varphi(z) = -i\frac{z-1}{z+1}$ be the Möbius transformation mapping $B(0,1) \to \mathbb{H}$ with $f(-1) = \infty$, $f(1) = 0$ and $f(i) = 1$. A map $f\colon \mathbb{H} \to \mathbb{H}$ is a conformal bijection if and only if $\varphi \circ f \circ \varphi^{-1}$ is a conformal bijection $B(0,1) \to B(0,1)$. This implies that f has to be a Möbius transformation.

The Möbius transformations that map $\mathbb{H} \to \mathbb{H}$ need to map $\mathbb{R} \to \mathbb{R}$ while preserving orientation. Any such map f can be specified by three distinct points $x_1, x_2, x_3 \in \mathbb{R}\cup\{\infty\}$ and requiring $f(x_1) = \infty$, $f(x_2) = 0$ and $f(x_3) = 1$, while making sure that x_1, x_2, x_3 go cyclically from left to right on the real axis. Suppose first that $x_1, x_2, x_3 \neq \infty$. The map is then given by

$$f(z) = \frac{(z - x_2)(x_3 - x_1)}{(z - x_1)(x_3 - x_2)} = \frac{az + b}{cz + d}$$

with

$$a = x_3 - x_1, \quad b = -x_2(x_3 - x_1), \quad c = x_3 - x_2, \quad d = -x_1(x_3 - x_2).$$

Note that

$$ad - bc = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1)$$

and this is positive if and only if $x_1 < x_2 < x_3$ or $x_3 < x_1 < x_2$ or $x_2 < x_3 < x_1$, which is the case if and only if the points are ordered cyclically from left to right. If one of the points, say x_3, equals ∞, then

$$f(z) = \frac{z - x_2}{z - x_1} = \frac{az + b}{cz + d}$$

with

$$a = 1, \quad b = -x_2, \quad c = 1, \quad d = -x_1,$$

and $ad - bc = x_2 - x_1$, which is positive if and only if $x_2 > x_1$, again meaning that x_1, x_2, ∞ are ordered cyclically from left to right. The cases $x_1 = \infty$ or $x_2 = \infty$ can be handled similarly. □

These results are useful for instance when one thinks about uniqueness in the Riemann mapping theorem.

Example 7.21. Suppose that $U \neq \mathbb{C}$ is simply connected and $f, g\colon U \to \mathbb{H}$ are two conformal bijections. Then $\varphi = f \circ g^{-1}$ is a conformal automorphism of $\mathbb{H}$, and hence a Möbius transformation. Thus, f and g are related by $f = \varphi \circ g$. ◆

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